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CHARACTERIZATIONS OF RARELY g-CONTINUOUS MULTIFUNCTIONS

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Abstract. In 1979, Popa [15] introduced the notion of rare continuity. Quite recently, the authors [3] introduced and investigated a new class of functions called rarely g-continuous functions as a generalization of both rare continuity and weak g-continuity [5]. In this paper, we introduce and study the new notion of upper (lower) rarely g-continuous multifunctions as a generalization of upper (lower) weakly continuous multifunctions [13].

1. Introduction

In 1979, Popa [14] introduced the notion of rare continuity as a generalization of weak continuity [10]. This notion has been further investigated by Long and Herrington [12] and Jafari [8] and [9]. Levine [11] introduced the concept of generalized closed sets of a topological space and a class of topological spaces called $T_{1/2}$-spaces. Dunham [6], Dunham and Levine [7] and Caldas [2] further studied some properties of generalized closed sets and $T_{1/2}$-spaces.

Quite recently, the authors [3] introduced and investigated a new class of functions called rarely g-continuous functions as a generalization of both rare continuity and weak g-continuity [5]. The purpose of the present paper is to offer the multifunction version of this type of continuity, i.e. the notion of rare g-continuity, as a generalization of rare continuous multifunctions [14] and weakly g-continuous multifunctions [4] and study some of its properties.

2. Preliminaries

Throughout this paper, X and Y are topological spaces. Recall that a rare set is a set R such that $Int(R) = \emptyset$. Levine [11] introduced the notion of g-closed sets: A set A in X is called g-closed if $Cl(A) \subset G$ whenever $A \subset G$ and G is open in X. The complement of a g-closed set is called g-open [11].

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The family of all g-open (resp. open) sets will be denoted by $GO(X)$ (resp. $O(X)$). We set $GO(X,x) = \{U \mid x \in U \in GO(X)\}$, $O(X,x) = \{U \mid x \in U \in O(X)\}$ and $O(X,A) = \{U \mid A \subset U \in O(X)\}$. The g-interior of A, denoted by $Int_g(A)$, is defined by $Int_g(A) = \bigcup\{U \in GO(X) \mid U \subset A\}$. Note that for any subsets A and B of a space X, $Int_g(A) \subset Int_g(B)$ if $A \subset B$.

Definition 1. A function $f : X \to Y$ is called:

i) weakly continuous [10] (resp. weakly-g-continuous [4]) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X,x)$ (resp. $U \in GO(X,x)$) such that $f(U) \subset Cl(G)$,
ii) rarely continuous [14] (resp. rarely g-continuous [3])if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in O(X,x)$ (resp. $U \in GO(X,x)$) such that $f(U) \subset G \cup R_G$,
iii) g-continuous [1] if the inverse image of every closed set in Y is g-closed in X.

Note that, every weakly continuous function is rarely continuous and every rarely continuous function is rarely g-continuous.

3. Upper (lower) rarely g-continuous multifunctions

We provide the following definitions which will be used in the sequel. Let $F : X \to Y$ be a multifunction. The upper and lower inverses of a set $V \subset Y$ are denoted by $F^+(V)$ and $F^-(V)$ respectively, that is,

$$F^+(V) = \{x \in X | F(x) \subset V\}$$

and

$$F^-(V) = \{x \in X | F(x) \cap V \neq \emptyset\}.$$

Definition 2. A multifunction $F : X \to Y$ is said to be

(i) upper rarely g-continuous (briefly $u.r.g.c$)at $x \in X$ if for each $V \in O(Y, F(x))$, there exist a rare set R_V with $V \cap Cl(R_V) = \emptyset$ and $U \in GO(X,x)$ such that $F(U) \subset V \cup R_V$,
(ii) lower rarely g-continuous (briefly $l.r.g.c$) at $x \in X$ if for each $V \in O(Y)$ with $F(x) \cap V \neq \emptyset$ there exist a rare set R_V with $V \cap Cl(R_V) = \emptyset$ and $U \in GO(X,x)$ such that $F(u) \cap (V \cup R_V) \neq \emptyset$ for every $u \in U$,
(iii) upper/lower rarely g-continuous if it is upper/lower rarely g-continuous at each point of X.

Definition 3. ([4]) A multifunction $F : X \to Y$ is said to be

(i) upper weakly g-continuous at $x \in X$ if for each $V \in O(Y, F(x))$, there exist $U \in GO(X,x)$ such that $F(U) \subset Cl(V)$,

(ii) lower weakly g-continuous at $x \in X$ if for each $V \in O(Y)$ with $F(x) \cap V \neq \emptyset$, there exists $U \in GO(X,x)$ such that $F(u) \cap Cl(V) \neq \emptyset$ for every $u \in U$,

(iii) upper/lower weakly g-continuous if it is upper/lower weakly g-continuous at each point of X.

Theorem 1. *The following statements are equivalent for a multifunction* $F : X \to Y$*:*

(i) *F is u.r.g.c at $x \in X$,*

(ii) *For each $V \in O(Y, F(x))$, there exists $U \in GO(X,x)$ such that* $\mathrm{Int}[F(U) \cap (Y - V)] = \emptyset$,

(iii) *For each $V \in O(Y, F(x))$, there exists $U \in GO(X,x)$ such that* $\mathrm{Int}[F(U)] \subset Cl(V)$.

Proof. $(i) \Rightarrow (ii)$: Let $V \in O(Y, F(x))$. By $F(x) \subset V \subset Int(Cl(V))$ and the fact that $Int(Cl(V)) \in O(Y)$, there exist a rare set R_V with $Int(Cl(V)) \cap Cl(R_V) = \emptyset$ and a g-open set $U \subset X$ containing x such that $F(U) \subset Int(Cl(V)) \cup R_V$. We have $Int[F(U) \cap (Y - V)] = Int(F(U)) \cap Int(Y - V) \subset Int(Cl(V) \cup R_V) \cap (Y - Cl(V)) \subset (Cl(V) \cup Int(R_V)) \cap (Y - Cl(V)) = \emptyset$.

$(ii) \Rightarrow (iii)$: Obvious.

$(iii) \Rightarrow (i)$: Let $V \in O(Y, F(x))$. Then, by (iii) there exists $U \in GO(X,x)$ such that $\mathrm{Int}[F(U)] \subset Cl(V)$. Thus $F(U) = [F(U) - \mathrm{Int}(F(U))] \cup \mathrm{Int}[F(U)] \subset [F(U) - \mathrm{Int}(F(U))] \cup Cl(V) = [F(U) - \mathrm{Int}(F(U))] \cup V \cup (Cl(V) - V) = [(F(U) - \mathrm{Int}(F(U)) \cap (Y - V)] \cup V \cup (Cl(V) - V)$. Put $P = (F(U) - \mathrm{Int}(F(U))) \cap (Y - V)$ and $G = Cl(V) - V$, then P and G are rare sets. Moreover, $R_V = P \cup G$ is a rare set such that $Cl(R_V) \cap V = \emptyset$ and $F(U) \subset V \cup R_V$. Hence F is *u.r.g.c.* □

Theorem 2. *The following are equivalent for a multifunction* $F : X \to Y$:

(i) *F is l.r.g.c at $x \in X$,*

(ii) *For each $V \in O(Y)$ such that $F(x) \cap V \neq \emptyset$ there exists a rare set R_V with $V \cap Cl(R_V) = \emptyset$ such that $x \in \mathrm{Int}_g(F^-(V \cup R_V))$,*

(iii) *For each $V \in O(Y)$ such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $Cl(V) \cap R_V = \emptyset$ such that $x \in$ $Int_g(F^-(Cl(V) \cup R_V))$,*

(iv) *For each $V \in RO(Y)$ such that $F(x) \cap V \neq \emptyset$, there exists a rare set R_V with $V \cap Cl(R_V) = \emptyset$ such that $x \in \mathrm{Int}_g(F^-(V \cup R_V))$.*

Proof. $(i) \Rightarrow (ii)$: Let $V \in O(Y)$ such that $F(x) \cap V \neq \emptyset$. By (i), there exist a rare set R_V with $V \cap Cl(R_V) = \emptyset$ and $U \in GO(X,x)$ such that $F(x) \cap (V \cup R_V) \neq \emptyset$ for each $u \in U$. Therefore, $u \in F^-(V \cup R_V)$ for each $u \in U$ and hence $U \subset F^-(V \cup R_V)$. Since $U \in GO(X,x)$, we obtain $x \in U \subset \mathrm{Int}_g(F^-(V \cup R_V))$.

(ii) $\Rightarrow$ *(iii)* : Let $V \in O(Y)$ such that $F(x) \cap V \neq \emptyset$. By (ii), there exists a rare set R_V with $V \cap Cl(R_V) = \emptyset$ such that $x \in \mathrm{Int}_g(F^-(V \cup R_V))$. We have $R_V \subset Y - V = (Y - Cl(V)) \cup (Cl(V) - V)$ and hence $R_V \subset [R_V \cap (Y - Cl(V))] \cup (Cl(V) - V)$. Now, put $P = R_V \cap (Y - Cl(V))$. Then P is a rare set and $P \cap Cl(V) = \emptyset$. Moreover, we have $x \in \mathrm{Int}_g(F^-(V \cup R_V)) \subset \mathrm{Int}_g(F^-(P \cup Cl(V)))$.

(iii) $\Rightarrow$ *(iv)* : Let V be any regular open set of Y such that $F(x) \cap V \neq \emptyset$. By (iii), there exists a rare set R_V with $Cl(V) \cap R_V = \emptyset$ such that $x \in \mathrm{Int}_g(F^-(Cl(V) \cup R_V))$. Put $P = R_V \cup (Cl(V) - V)$, then P is a rare set and $V \cap Cl(P) = \emptyset$. Moreover, we have $x \in \mathrm{Int}_g(F^-(Cl(V) \cup R_V)) = \mathrm{Int}_g(F^-(R \cup (Cl(V) - V) \cup V)) = \mathrm{Int}_g(F^-(P \cup V))$.

(iv) $\Rightarrow$ *(i)* : Let $V \in O(Y)$ such that $F(x) \cap V \neq \emptyset$. Then $F(x) \cap \mathrm{Int}(Cl(V)) \neq \emptyset$ and $\mathrm{Int}(Cl(V))$ is regular open in Y. By (iv), there exists a rare set R_V with $V \cap Cl(R_V) = \emptyset$ such that $x \in \mathrm{Int}_g(F^-(V \cup R_V))$. Therefore, there exists $U \in GO(X, x)$ such that $U \subset F^-(V \cup R_V)$; hence $F(u) \cap (V \cup R_V) \neq \emptyset$ for each $u \in U$. This shows that F is lower rarely g-continuous at x. □

Corollary 1. ([[3], Theorem 2]) *The following statements are equivalent for a function* $f : X \to Y$:

(i) f *is rarely* g*-continuous at* $x \in X$,
(ii) *For* $V \in O(Y, f(x))$, *there exists* $U \in GO(X, x)$ *such that* $\mathrm{Int}[f(U) \cap (Y - V)] = \emptyset$,
(iii) *For each* $V \in O(Y, f(x))$, *there exists* $U \in GO(X, x)$ *such that* $\mathrm{Int}[f(U)] \subset Cl(V)$.

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