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POINTWISE PRODUCTS OF UNIFORMLY CONTINUOUS FUNCTIONS

SAM B. NADLER, JR.

ABSTRACT. The problem of characterizing the metric spaces on which the pointwise product of any two uniformly continuous real - valued functions is uniformly continuous is investigated. A sufficient condition is given; furthermore, the condition is shown to be necessary for certain types of metric spaces, which include those with no isolated point and all subspaces of Euclidean spaces. It is not known if the condition is always necessary.

1. INTRODUCTION

The question of when pointwise products of uniformly continuous real-valued functions on a metric space must be uniformly continuous has been investigated in several papers (e.g., [1], [2], [6]). Almost all results focus on special types of uniformly continuous functions or involve complicated conditions on the metric. An exception is Atsuji's paper [1]: for the case of *connected* metric spaces, Atsuji gives a necessary condition for pointwise products of uniformly continuous real - valued functions to be uniformly continuous ([1], Theorem 3); in fact, the condition is also sufficient (as we note following Theorem 4.2). Nevertheless, as far as we know, there is no complete characterization of those metric spaces, even subspaces of the real line $\mathbb{R}^1$, on which the pointwise product of any two uniformly continuous real-valued functions is uniformly continuous. This is surprising to us in view of the vast literature on rings and semigroups of continuous functions. We state the problem for convenient reference; for brevity, we let $\mathcal{U}(X)$ denote the set of all uniformly continuous real - valued functions on a metric space (X, d).

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Characterization Problem. *Determine intrinsic conditions that characterize those metric spaces for which $\mathcal{U}(X)$ is closed under pointwise product.*

We give a general sufficient condition for $\mathcal{U}(X)$ to be closed under pointwise products (we do not know if the condition is also necessary); then we show the condition is necessary, thus solving the Characterization Problem, for two types of spaces – metric spaces with no isolated point, and subspaces of certain types of metric spaces (called W - B spaces) which include all Euclidean spaces. Our main results are Theorems 3.1, 4.2 and 5.4 and Corollary 5.5.

2. Terminology and notation

We present most of the terminology and notation that we use; we include brief comments about notions we define. We do not include definitions that we feel are standard enough to be omitted.

Twin sequences are a central idea in the paper. We say that two sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, in a metric space (X, d) are *twin sequences* provided that

$$\lim_{n\to\infty} d(a_n, b_n) = 0 \text{ and } a_n \neq b_n \text{ for all } n.$$

A metric space is said to be *uniformly isolated* provided that there is a $\delta > 0$ such that any two points of the space are more than δ apart.

A metric space is called a *uniformly continuous set* (abbreviated U.C. set) provided that all continuous real - valued functions on the space are uniformly continuous [5]. We give a characterization of U.C. sets in Theorem 5.2 that is particularly useful for us. For other results about U.C. sets, see for example ([1], Theorem 1), [3], [7] [8, p. 368] and [9].

A *W - B space* is a metric space in which all closed and bounded subsets are compact (the terminology is from [7]).

We note the following connection between notions we have defined: *A subspace Z of a W - B space is a U.C. set if and only if Z is the union of a compact set and a uniformly isolated set* (Theorem 4 of [7]).

Next, we define the notion of finite chainability. The importance of the notion is that it characterizes those metric spaces on which every uniformly continuous real - valued function is bounded ([1], Theorem 2).

Let (X, d) be a metric space. An ϵ *- chain in X from x to y of length m* is a finite sequence

$$x_0 = x, x_1, x_2, \ldots, x_m = y$$

such that $d(x_{i-1}, x_i) < \epsilon$ for all $i = 1, 2, \ldots, m$. We say that (X, d) is *finitely chainable* provided that for each $\epsilon > 0$, there are finitely many points $p_1, p_2, \ldots, p_n \in X$ and a positive integer m such that there is an ϵ - chain in

X of length m from any point x of X to one of the points $p_1, p_2, \ldots, p_n$ ([1], p. 14).

Finite chainability is a generalization of totally bounded (*totally bounded* means $m = 1$ in the definition of finitely chainable). The unit ball B in any infinite dimensional Banach space is not totally bounded but is finitely chainable (use points on radial lines from the center of B to form the required chains). Finitely chainable spaces are always bounded; for subspaces of W– B spaces, finite chainability is equivalent to being bounded.

If (X, d) is a metric space and A and B are nonempty subsets of X, then

$$d(A, B) = \inf \big\{ d(a, b) : a \in A \text{ and } b \in B \big\}.$$

We denote the restriction of a map f to a subspace A by $f|A$. We use $\overline{E}$ to denote the closure of E in the given (largest) space.

We assume that all subsets of metric spaces have the subspace metric and that the real line $\mathbb{R}^1$ has its usual metric.

3. A sufficient condition for metric spaces

In Theorem 3.1 we give a general class of metric spaces for which $\mathcal{U}(X)$ is closed under pointwise product. We do not know if the converse of the theorem is true; however, we show in section 4 that the converse is true for metric spaces with no isolated point (Theorem 4.2), and we show in section 5 that the converse holds for subspaces of W - B spaces (Theorem 5.4). Other partial converses of Theorem 3.1 are Theorem 5.7 and Corollary 5.8.

Theorem 3.1. *Let (X, d) be a metric space. If X is the union of a finitely chainable subspace F and a uniformly isolated subspace I, then $\mathcal{U}(X)$ is closed under pointwise product.*

Proof. Let $f, g : X \to \mathbb{R}^1$ be uniformly continuous functions.

Since I is uniformly isolated, there exists $\delta_1 > 0$ such that for all $x, y \in I$ such that $x \neq y$,

$$d(x, y) > \delta_1.$$

Note that the restricted maps $f|F$ and $g|F$ are uniformly continuous. Thus, since F is finitely chainable, $f|F$ and $g|F$ are bounded ([1], Theorem 2). Hence, there is an $M > 0$ such that

$$|f(x)| < M \text{ and } |g(x)| < M \quad \text{for all } x \in F.$$

Also, by uniform continuity, there is a $\delta_2 > 0$ such that if $x, y \in X$ and $d(x, y) < \delta_2$, then

$$|f(x) - f(y)| < 1 \text{ and } |g(x) - g(y)| < 1.$$

Thus, for each $x \in F$,

$$|f(y)| < M+1 \text{ and } |g(y)| < M+1 \text{ for all } y \in B_d(x, \delta_2).$$

Next, by uniform continuity, there exists $\delta_3 > 0$ such that if $x, y \in X$ and $d(x,y) < \delta_3$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2(M+1)} \text{ and } |g(x) - g(y)| < \frac{\epsilon}{2(M+1)}.$$

Let

$$\delta = \min\{\delta_1, \delta_2, \delta_3\}.$$

Now, let $x, y \in X$ such that $x \neq y$ and $d(x,y) < \delta$. Since $\delta \leq \delta_1$, x and y are not both in I; hence, we assume without loss of generality that $x \in F$. Thus, $|f(x)| < M$ and, since $y \in B_d(x, \delta_2)$, $|g(y)| < M+1$. Therefore,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= \big|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)\big| \\ &\leq |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\ &< M\frac{\epsilon}{2(M+1)} + (M+1)\frac{\epsilon}{2(M+1)} < \epsilon. \end{aligned}$$

This proves that $f \cdot g$ is uniformly continuous. □

4. Characterization when X has no isolated point

We prove that the converse of Theorem 3.1 is true when X has no isolated point.

First we need a lemma; the lemma is probably known but does not seem to be stated in the literature, so we include a proof of the lemma.

Lemma 4.1. *Let (X, d) be a metric space, and let A and B be nonempty subsets of X such that $d(A, B) > 0$. Define $\varphi : X \to \mathbb{R}^1$ by*

$$\varphi(x) = \frac{d(x,A)}{d(x,A) + d(x,B)} \text{ for all } x \in X.$$

Then φ is uniformly continuous.

Proof. Let $\epsilon > 0$. Let $\delta = \epsilon \cdot d(A,B)$. Note that $\delta > 0$ by our assumption about A and B.

Let $p, q \in X$ such that $d(p,q) < \delta$. Then

$$\begin{aligned} |\varphi(p) - \varphi(q)| &= \left|\frac{d(p,A)}{d(p,A) + d(p,B)} - \frac{d(q,A)}{d(q,A) + d(q,B)}\right| \\ &= \left|\frac{d(p,A)d(q,B) - d(q,A)d(p,B)}{[d(p,A) + d(p,B)][d(q,A) + d(q,B)]}\right| \\ &= \left|\frac{d(p,A)d(q,B) - d(p,A)d(p,B) + d(p,A)d(p,B) - d(q,A)d(p,B)}{[d(p,A) + d(p,B)][d(q,A) + d(q,B)]}\right| \end{aligned}$$

$$= \left| \frac{d(p,A)[d(q,B) - d(p,B)] + d(p,B)[d(p,A) - d(q,A)]}{[d(p,A) + d(p,B)][d(q,A) + d(q,B)]} \right|$$
$$\leq \frac{d(p,A)\,|d(q,B) - d(p,B)| + d(p,B)\,|d(p,A) - d(q,A)|}{[d(p,A) + d(p,B)][d(q,A) + d(q,B)]}$$
$$\leq \frac{d(p,A)\,|q-p| + d(p,B)\,|p-q|}{[d(p,A) + d(p,B)][d(q,A) + d(q,B)]} \text{ (by [4], (6'), p. 210)}$$
$$= \frac{|p-q|}{d(q,A) + d(q,B)} \leq \frac{|p-q|}{d(A,B)} < \frac{\delta}{d(A,B)} = \epsilon.$$

□

Theorem 4.2. *Let (X,d) be a metric space with no isolated point. Then $\mathcal{U}(X)$ is closed under pointwise product if and only if (X,d) is finitely chainable.*

Proof. The fact that finite chainability is sufficient is due to Theorem 3.1.

Conversely, assume that (X,d) is not finitely chainable. Then, by Theorem 2 of [1], there is a uniformly continuous unbounded function $g : X \to \mathbb{R}^1$. Hence, there is a sequence $\{a_n\}_{n=1}^{\infty}$ in X such that $g(a_n) \geq n$ for each n and such that, without loss of generality, $a_n \neq a_m$ for all $n \neq m$.

By the continuity of g, no subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to a point of X; in particular, no subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to a_j for any j. Thus, since $a_n \neq a_m$ for all $n \neq m$, there are mutually disjoint open neighborhoods U_n of the points a_n in X. Therefore, since a_n is a limit point of X, there is, for each n, a point $b_n \in U_n$ such that the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are twin sequences and $g(b_n) \geq \frac{n}{2}$.

For each n, let
$$A_n = \{a_n\} \cup (\overline{U}_n - U_n),$$
and define $f : X \to \mathbb{R}^1$ by
$$f(x) = \begin{cases} \dfrac{d(x,A_n)}{n[d(x,A_n) + d(x,b_n)]}, & \text{if } x \in \overline{U}_n \text{ for some } n \\ 0, & \text{otherwise.} \end{cases}$$

We prove that f is uniformly continuous. Let $\epsilon > 0$. Fix N such that $\frac{1}{N} < \epsilon$. If $x, y \in \cup_{n=N}^{\infty} U_n$, then $f(x), f(y) \in [0, \frac{1}{N}]$; hence,
$$|f(x) - f(y)| \leq \frac{1}{N} < \epsilon \text{ for } x, y \in \cup_{n=N}^{\infty} U_n;$$
furthermore, f is uniformly continuous on $\overline{U}_n$ for each $n < N$ by Lemma 4.1 (since $d(A_n, b_n) > 0$ for each n). Therefore, since $f(\overline{U}_n - U_n) = 0$ for each n, it follows easily that f is uniformly continuous on all of X.

Finally, $g \cdot f$ is not uniformly continuous since $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are twin sequences such that

$$g(a_n)f(a_n) = 0 \text{ and } g(b_n)f(b_n) \geq \frac{n}{2} f(b_n) = \frac{1}{2}.$$

□

Theorem 4.2 is a generalization of a theorem that should be credited to Atsuji – namely, *For a connected metric space* $(X, d), \mathcal{U}(X)$ *is closed under pointwise product if and only if* (X, d) *is finitely chainable.* Atsuji proved the necessity of being finitely chainable ([1], Theorem 3), but did not note the sufficiency even though it follows easily from one of his theorems ([1], Theorem 2) and from the elementary fact that the pointwise product of two bounded uniformly continuous real-valued functions is uniformly continuous.

5. Characterization theorems for W-B spaces

Recall that a *W-B space* is a metric space in which all closed and bounded subsets are compact. We characterize the subspaces Y of W-B spaces for which $\mathcal{U}(Y)$ is closed under pointwise product (Theorem 5.4 and Corollary 5.5). Our most descriptive characterizations are the equivalences of (1) and (2) in Theorem 5.4 and Corollary 5.5; we include the equivalences of (1) and (3) in order to specifically state a satisfying connection between (1) and uniformly continuous sets in W-B spaces (the equivalence of (1) and (3) does not hold in general, as we note in Example 5.6).

First, in Theorem 5.2, we characterize U.C. sets for metric spaces in terms of twin sequences. We use Theorem 5.2 in the proof of Theorem 5.4.

We can deduce Theorem 5.2 from the equivalence of (1) and (7) in Theorem 1 of [1]. However, the proof of Theorem 1 of [1] involves verifying a lengthy chain of implications and several verifications are somewhat complicated; thus, we prefer in the interest of completeness to include a short, simple proof of Theorem 5.2 that is independent of [1].

Lemma 5.1. *Let* (X, d) *be a metric space. If* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are twin sequences in* X *that have no subsequences that converge to a point of* X*, then there are twin subsequences,* $\{a_{n_i}\}_{i=1}^{\infty}$ *and* $\{b_{n_i}\}_{i=1}^{\infty}$ *, such that* $a_{n_i} \neq a_{n_j}$ *and* $b_{n_i} \neq b_{n_j}$ *for all* $i \neq j$ *and* $a_{n_i} \neq b_{n_j}$ *for all* i *and* j*.*

Proof. Let $n_1 = 1$ and note from the definition of twin sequence that $a_{n_1} \neq b_{n_1}$.

Assume inductively that we have defined $n_1 < n_2 < \cdots < n_k$ such that for all $i, j \leq k$, $a_{n_i} \neq a_{n_j}$ and $b_{n_i} \neq b_{n_j}$ when $i \neq j$ and $a_{n_i} \neq b_{n_j}$.

Then, since each term of the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ is repeated at most finitely many times, there exists n_{k+1} such that for all $i \leq k$,

$$a_{n_{k+1}} \neq a_{n_i},\ b_{n_{k+1}} \neq b_{n_i} \ \text{ and } \ a_{n_{k+1}} \neq b_{n_i}.$$

Note that $a_{n_{k+1}} \neq b_{n_{k+1}}$ by the definition of twin sequence.

Then, by induction, we have defined the required subsequences $\{a_{n_i}\}_{i=1}^{\infty}$ and $\{b_{n_i}\}_{i=1}^{\infty}$. □

Theorem 5.2. *Let (X, d) be a metric space. Then (X, d) is a U.C. set if and only if all twin sequences in X have subsequences that converge to a point of X.*

Proof. Assume that (X, d) is not a U.C. set. Then there is a continuous function $f : X \to \mathbb{R}^1$ such that f is not uniformly continuous. Thus, there exists $\epsilon > 0$ such that for each $n = 1, 2, \ldots$, there are points x_n and y_n in X for which

$$d(x_n, y_n) < \frac{1}{n} \ \text{ and } \ |f(x_n) - f(y_n)| \geq \epsilon.$$

We see that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are twin sequences that have no subsequences converging to a point of X (if $\lim_{i\to\infty} x_{n_i} = p$, then $\lim_{i\to\infty} y_{n_i} = p$ and, hence, f is not continuous at p, a contradiction).

Conversely, assume there are twin sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, in X that have no subsequences converging to a point of X. By Lemma 5.1, we assume without loss of generality that $a_n \neq a_m$ and $b_n \neq b_m$ for all $n \neq m$ and that $a_n \neq b_m$ for all n and m. Then, letting

$$Y = \{a_n : n = 1, 2, \ldots\} \cup \{b_n : n = 1, 2, \ldots\},$$

the following formula defines a function $g : Y \to \mathbb{R}^1$:

$$g(y) = \begin{cases} n, & \text{if } y = a_n \\ n+1, & \text{if } y = b_n. \end{cases}$$

Since Y has no limit point, g is continuous and Y is closed in X. Thus, g can be extended to a continuous function $h : X \to \mathbb{R}^1$ [4, pp. 127 - 128]. Since $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are twin sequences, g and, hence, h is not uniformly continuous. Therefore, X is not a U.C. set. □

Lemma 5.3. *Let (X, d) be a metric space. If $\mathcal{U}(X)$ is closed under pointwise product, then twin sequences in X are bounded.*

Proof. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are unbounded twin sequences in X. By going to subsequences if necessary, we assume without loss of generality that

$$d(a_1, \{a_n, b_n\}) \geq n \ \text{ for all } \ n > 1$$

and, by Lemma 5.1, that $a_n \neq a_m$ and $b_n \neq b_m$ for all $n \neq m$ and $a_n \neq b_m$ for all n and m.

It follows that for all n, there are open neighborhoods U_n of $\{a_n, b_n\}$ in X such that

$$U_n \cap U_m = \emptyset \text{ when } n \neq m.$$

Now, define A_n and f exactly as in the proof of Theorem 4.2, where it is shown that f is uniformly continuous.

Next, define $g : X \to \mathbb{R}^1$ by

$$g(x) = d(a_1, x) \text{ for all } x \in X.$$

A simple argument using the triangle inequality shows that g is nonexpansive, hence uniformly continuous.

Finally, $f \cdot g$ is not uniformly continuous since $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are twin sequences such that

$$f(a_n)g(a_n) = 0 \text{ and } f(b_n)g(b_n) = \frac{1}{n}d(a_1, b_n) \geq 1 (n > 1).$$

□

We are ready to prove the main results of the section. Note that bounded sets in W - B spaces are totally bounded and, hence, are finitely chainable. Thus, the fact that (1) implies (2) in the following theorem shows that the converse of Theorem 3.1 is true for subspaces of W - B spaces.

Theorem 5.4. *Let (X, d) be a W - B space, and let $Y \subset X$. Then the following are equivalent:*

(1) *$\mathcal{U}(Y)$ is closed under pointwise product;*
(2) *Y is the union of a bounded set and a uniformly isolated set.*
(3) *$\overline{Y}$ is a U.C. set.*

Proof. We first prove (1) implies (3). Assume (1). Then, by Lemma 5.3, twin sequences in Y are bounded. Hence, twin sequences in $\overline{Y}$ are bounded (as follows: let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be twin sequences in $\overline{Y}$; then, for each n, there are c_n and d_n in Y within $\frac{d(a_n, b_n)}{3}$ of a_n and b_n, respectively; we see that $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are twin sequences in Y and, hence, are bounded; therefore, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded). Thus, since closed and bounded subsets of $\overline{Y}$ are compact, it follows that all twin sequences in $\overline{Y}$ have subsequences that converge to a point of $\overline{Y}$. Therefore, by Theorem 5.2, $\overline{Y}$ is a U.C. set. This proves (1) implies (3).

Next, assume (3). Then, by Theorem 4 of [7], $\overline{Y}$ is the union of a compact set C and a uniformly isolated set I. Hence, Y is the union of the bounded set $C \cap Y$ and the uniformly isolated set $I \cap Y$. This proves (3) implies (2).

Finally, (2) implies (1) by Theorem 3.1 (since, as noted above, bounded sets in Y are totally bounded, hence finitely chainable). □

Corollary 5.5. *If (X, d) is a W - B space, then the following are equivalent:*

(1) *$\mathcal{U}(X)$ is closed under pointwise product;*
(2) *X is the union of a compact set and a uniformly isolated set;*
(3) *(X, d) is a U.C. set.*

Proof. The corollary follows immediately from Theorem 5.4 ((1) implies (2) by Theorem 5.4 since the closure of a bounded set in a W - B space is compact). □

We give an example to show that assuming that (X, d) is a W - B space is necessary for (1) to imply (3) in Theorem 5.4 and Corollary 5.5 (we will see in Theorem 5.7 that (1) always implies (2) in Theorem 5.4).

Example 5.6. Let B be the unit ball in an infinite dimensional Banach space. Then B is finitely chainable (as shown in section 2); thus, by Theorem 3.1, B satisfies (1) of Theorem 5.4 and Corollary 5.5. However, it is easy to see that B does not satisfy (3) of Theorem 5.4 and Corollary 5.5.

Our next theorem shows that (1) implies (2) in Theorem 5.4 without assuming (X, d) is a W - B space. The theorem also provides a partial converse to Theorem 3.1. The corollary that follows the theorem is a characterization in the setting of spaces that satisfy a condition similar to the condition that defines W - B spaces.

Theorem 5.7. *Let (X, d) be a metric space, and let L denote the set of all limit points of X. If $\mathcal{U}(X)$ is closed under pointwise product, then L is bounded and the complement of some bounded neighborhood of L in X is uniformly isolated.*

Proof. For each $n = 1, 2, \dots$, let

$$B_n = \{x \in X : d(x, L) < n\}.$$

If L is unbounded or if $X - B_n$ is not uniformly isolated for any n, then, in either case, we can easily obtain unbounded twin sequences in X; this contradicts Lemma 5.3. Hence, L is bounded and $X - B_k$ is uniformly isolated for some k. Since L is bounded, B_k is bounded; clearly, B_k is a neighborhood of L in X. □

Corollary 5.8. *Let (X, d) be a metric space in which all bounded subspaces are finitely chainable. Then $\mathcal{U}(X)$ is closed under pointwise product if and only if X is the union of a finitely chainable subspace and a uniformly isolated subspace.*

Proof. The corollary is due to Theorem 5.7 and Theorem 3.1. □

Regarding Theorem 5.7, the following example shows that the complement of some bounded neighborhood of L in X may *not* be uniformly isolated:

Example 5.9. Let X be the subspace of the Euclidean plane $\mathbb{R}^2$ consisting of the points $(\frac{1}{n}, 0)$ and $(\frac{1}{n}, \frac{1}{m})$ for $n, m = 1, 2, \ldots$. Since X is a bounded set in the W-B space $\mathbb{R}^2$, $\mathcal{U}(X)$ is closed under pointwise product by Theorem 5.4. Let

$$U = \{(x, y) \in X : y < x\};$$

then U is a bounded neighborhood of $L = \{(\frac{1}{n}, 0) : n = 1, 2, \ldots\}$ in X such that $X - U$ is not uniformly isolated.

6. A reduction for the characterization problem

We conclude with a theorem that is easy to prove but that might be useful to be aware of in working on the Characterization Problem: the theorem reduces the Problem to complete spaces.

We let $\widehat{X}$ denote the completion of (X, d). We do not distinguish between X and the natural copy of X that is isometrically embedded in $\widehat{X}$ as a dense subset of $\widehat{X}$.

Theorem 6.1. *Let (X, d) be a metric space. Then $\mathcal{U}(X)$ is closed under pointwise product if and only if $\mathcal{U}(\widehat{X})$ is closed under pointwise product.*

Proof. Assume that $\mathcal{U}(X)$ is closed under pointwise product, and let $f, g : \widehat{X} \to \mathbb{R}^1$ be uniformly continuous. Then $(f|X)\cdot(g|X) = (f\cdot g)|X$ is uniformly continuous. Therefore, since X is dense in $\widehat{X}$, $f \cdot g$ is uniformly continuous.

Conversely, assume that $\mathcal{U}(\widehat{X})$ is closed under pointwise product, and let $f, g : X \to \mathbb{R}^1$ be uniformly continuous. Then f and g can be extended (uniquely) to uniformly continuous maps $F, G : \widehat{X} \to \mathbb{R}^1$ (10.66 of [10, p. 324]). Hence, $F \cdot G$ is uniformly continuous. Therefore, since $(F \cdot G)|X = f \cdot g$, $f \cdot g$ is uniformly continuous. □

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West Virginia University
P. O. Box 6310
Morgantown, WV 26506-6310
USA
E–mail: nadler@math.wvu.edu