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AN EXTENSION OF TWO RESULTS OF HARDY

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Abstract. In this paper we extend two results of Hardy, dealing with regular weighted mean matrices, to coregular factorable matrices.

A weighted mean matrix is a lower triangular infinite matrix with nonzero entries $p_k/P_n, 0 \leq k \leq n$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$, and $P_n := \sum_{k=0}^{n} p_k$. A factorable matrix is a lower triangular matrix with nonnegative entries $a_n b_k, 0 \leq k \leq n$. Obviously a weighted mean matrix is a special case of a factorable matrix obtained by setting $b_k = p_k, a_n = 1/P_n$.

An infinite matrix is said to be regular if it is limit-preserving over c, the space of convergent sequences. Necessary and sufficient conditions for a matrix $A = (a_{nk})$ to be regular are the well-known Silverman-Toeplitz conditions:

(i) $\|A\| := \sup_n \sum_k |a_{nk}| < \infty$,
(ii) $\lim_n a_{nk} = 0$ for each k,
(iii) $\lim_n \sum_k a_{nk} = 1$.

An infinite matrix A is said to be conservative if $A : c \to c$. The Silverman-Toeplitz conditions for a matrix to be conservative are

(i′) $\|A\| < \infty$,
(ii′) α_k exists for each k, where $\alpha_k := \lim_n a_{nk}$,
(iii′) $t := \lim_n t_n$ exists, where $t_n := \sum_k a_{nk}$.

A conservative matrix is called coregular if $\chi(A) := t - \sum_k \alpha_k \neq 0$.

It is a straightforward exercise to verify that a nonnegative conservative factorable matrix $A = (a_n b_k)$ is coregular if and only if

$$\lim_n a_n = 0 \quad \text{and} \quad t = \lim_n a_n B_n \neq 0,$$

where $B_n := \sum_{k=0}^{n} b_k$. A weighted mean matrix is regular if and only if $\lim P_n = \infty$.

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Given an infinite matrix A and a sequence x, $A_n(x) := \sum_k a_{nk} x_k$. The convergence domain of A, written c_A, is defined by

$$c_A = \{x : \lim_n A_n(x) \text{ exists}\}.$$

Theorem 1. *Let $A = (a_n b_k)$ be a nonnegative coregular factorable matrix. Then $c_A = c$ if and only if*

$$\liminf \frac{b_{n+1}}{B_n} > 0. \tag{1}$$

Proof. Note that, if $a_n = 0$ for an infinite number of values of n, say $\{n_i\}$, then A is not coregular, since then $t_{n_i} = 0$, which contradicts $t \neq 0$.

If $b_k = 0$ for an infinite sequence, say $\{k_j\}$, then, for the sequence x defined by $x_n = 0$ for $n \neq k_j, x_{k_j} = 1$, $A_n(x) = 0$ and hence $c_A \neq c$. Thus, if $c_A = c$, then there are only a finite number of values of n for which $a_n = 0$ or $b_n = 0$. On the other hand, if (1) holds, then clearly $b_n = 0$ for only a finite number of values of n. Thus the hypotheses of Theorem 1 ensure that there are only a finite number of values of n for which a_n or b_n is zero.

We now define a new factorable matrix D by

$$c_n = \begin{cases} a_n, & \text{if} \quad a_n \neq 0, \\ 1, & \text{if} \quad a_n = 0, \end{cases} \qquad d_n = \begin{cases} b_n, & \text{if} \quad b_n \neq 0, \\ 1, & \text{if} \quad b_n = 0. \end{cases}$$

Since there are only a finite number of values of n for which $a_n = 0, D$ is coregular since A is. Let N denote the largest value of n for which $a_n = 0$. Then, with $K := \{k_1, k_2, \ldots, k_M\}$, where $b_{k_1} = b_{k_2} = \cdots = b_{k_M} = 0$, for $n > N$,

$$\begin{aligned} D_n(x) &= \sum_{k=0}^{n} c_n d_k x_k = \sum_{k=0}^{n} a_n b_k x_k + \sum_{k \in K} a_n b_k x_k \\ &= A_n(x) + a_n \sum_{k \in K} b_k x_k, \end{aligned}$$

which implies that $c_A = c_D$.

Suppose that $\liminf a_{n-1} b_n = 0$. Then there exists a subsequence $\{n_i\}$ of $\mathbb{N}$ such that $\lim_i a_{n_i - 1} b_{n_i} = 0$. But

$$a_{n_i - 1} b_{n_i} = a_{n_i - 1} B_{n_i - 1} \frac{b_{n_i}}{B_{n_i - 1}},$$

and $\lim_i a_{n_i - 1} B_{n_i - 1} = t \neq 0$, which implies that

$$\liminf \frac{b_{n_i}}{B_{n_i - 1}} = 0,$$

contradicting (1). Therefore (1) implies that

$$\liminf a_{n-1} b_n > 0. \tag{2}$$

Now suppose that $\liminf a_n b_n = 0$. Then there exists a subsequence $\{n_j\}$ of $\mathbb{N}$ such that $\lim_j a_{n_j} b_{n_j} = 0$. Since

$$a_{n_j} b_{n_j} = a_{n_j} B_{n_j} \frac{b_{n_j}}{B_{n_j}},$$

and $\lim_j a_{n_j} B_{n_j} = t \neq 0$, it follows that

$$\lim_j \frac{b_{n_j}}{B_{n_j}} = 0.$$

But, assuming that $n_j \notin K$,

$$\frac{B_{n_j}}{b_{n_j}} = \frac{B_{n_j-1} + b_{n_j}}{b_{n_J}} = \frac{B_{n_j-1}}{b_{n_j}} + 1,$$

and hence $\lim_j b_{n_j}/B_{n_j} = 0$, again contradicting (1). Therefore (1) also implies that

$$\liminf a_n b_n \neq 0. \tag{3}$$

Since only a finite number of the a_n can be zero, conditions (2) and (3) imply that

$$\liminf c_{n-1} d_n \neq 0 \quad \text{and} \quad \liminf c_n d_n \neq 0. \tag{4}$$

A triangle is a lower triangular matrix with all of the main diagonal entries nonzero. Since D is a factorable triangle, it has a unique two sided inverse D^{-1} and D^{-1} is bidiagonal with entries

$$d_{nn}^{-1} = \frac{1}{c_n d_n}, \quad d_{n,n-1}^{-1} = -\frac{1}{c_{n-1} d_n},$$

and $d_{nk}^{-1} = 0$ otherwise. (See, e.g., Lemma 2.1 of [1].)

If $c_A = c$ then $c_D = c$ so that D^{-1} is conservative and the row norm condition implies that (4) must hold. Writing, for sufficiently large n,

$$a_n b_{n+1} = a_n B_n \frac{b_{n+1}}{B_n},$$

we see that (2) implies that (1) must hold by coregularity.

Conversely, if (1) holds, then (4) holds as above, and Theorem 3 of [8] gives the fact that D^{-1} is conservative. Hence $c_D = c$ and so $c_A = c$. □

Corollary 1. *Let A be a regular weighted mean matrix. Then $c_A = c$ if and only if*

$$\liminf \frac{p_n}{P_{n-1}} > 0. \tag{5}$$

Proof. Since a weighted mean matrix is a factorable matrix with $b_n = p_n, a_n = 1/P_n$, condition (1) becomes condition (5). $p_n/P_{n-1} \geq \theta/2$, and $p_n/P_n \geq \theta/(2+\theta)$ for all n sufficiently large. Hence $\liminf p_n/P_n > 0$. □

Corollary 1 is Theorem 4 of [2].

Historical Note. Corollary 2 was originally proved by Cesáro [3] in 1888, and was rediscovered by Hardy [4] in 1907.

Corollary 2. *Let A be a regular weighted mean matrix. Then $c_A = c$ if and only if*

$$\liminf \frac{P_{n+1}}{P_n} = 1 + \delta > 1, \quad \text{for some} \quad \delta > 0. \tag{6}$$

Proof. We may write

$$\frac{p_{n+1}}{P_n} = \frac{P_{n+1}}{P_n} - 1,$$

which implies that

$$\liminf \frac{p_{n+1}}{P_n} = \liminf \frac{P_{n+1}}{P_n} - 1 = \delta > 0.$$

Thus (6) is equivalent to (5), and the result follows from Corollary 1. □

Corollary 3. *Let A be a regular weighted mean matrix. Then, if*

$$\frac{P_{n+1}}{P_n} \geq 1 + \delta, \quad \text{for some} \quad \delta > 0, \tag{7}$$

$c_A = c$.

Proof. The result follows from Corollary 2, since (7) implies (6), with a possibly different δ. □

Corollary 3 is Theorem 15 of [5].

Theorem 2. *Let A be a regular factorable matrix. If*

$$\lim \frac{a_n}{a_{n-1}} = 1, \tag{8}$$

then $x \in c_A$ implies that $x_n = o(1/a_n b_n)$.

Proof. Define

$$u_n = a_n \sum_{k=0}^{n} b_k x_k.$$

But, for sufficiently large n,

$$\frac{u_n}{a_n} - \frac{u_{n-1}}{a_{n-1}} = b_n x_n,$$

which, in turn implies that

$$u_n - \left(\frac{a_n}{a_{n-1}}\right) u_{n-1} = a_n b_n x_n,$$

from which (8) follows. □

Corollary 4. *Let A be a regular weighted mean matrix. If*

$$\lim \frac{P_{n-1}}{P_n} = 1, \tag{9}$$

then $x \in c_A$ implies that $x_n = o(P_n/p_n)$.

Proof. With $a_n = 1/P_n$, (9) is equivalent to (8). □

Corollary 4 is Theorem 13 of [5].

Theorem 2 also provides a limitation theorem for Cesáro summability. The Cesáro matrix of order one, C, is a weighted mean matrix with $p_n = 1$ for all n. Thus $a_n = 1/(n+1)$, so that condition (6) is automatically satisfied. One then obtains $x_n = o(n)$, proving the following result.

Corollary 5. *If a sequence $x \in c_C$, then $x_n = o(n)$.*

Let $\{\lambda_n\}$ be a sequence satisfying

$$0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

such that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

For any sequence $\{\mu_n\}$, an H-J generalized Hausdorff matrix is a lower triangular matrix with entries

$$h_{nk} = \begin{cases} \lambda_{k+1} \ldots \lambda_n [\mu_k . \ldots, \mu_n], & 0 \leq k \leq n \\ 0, \quad k > n, \end{cases}$$

where it is understood that $\lambda_{k+1} \ldots \lambda_n = 1$ when $k = n$.

Hausdorff [6] made this definition for $\lambda_0 = 0$, and Jakimovski [7] extended it to the cases in which $\lambda_0 > 0$. If $\lambda_n = n$, then the definition reduces to that of an ordinary Hausdorff matrix. It is therefore reasonable to call such generalized Hausdorff matrices H-J matrices.

An H-J matrix is conservative if and only if the $\{\mu_n\}$ have the representation

$$\mu_n = \int_0^1 t^{\lambda_n} d\chi(t),$$

where $\chi(t)$ is a function of bounded variation over [0,1].

The H-J analogue of the Cesáro matrix of order one has

$$\mu_n = \int_0^1 t^{\lambda_n} dt = \frac{1}{\lambda_n + 1},$$

and the nonzero entries are

$$h_{nk} = \frac{\lambda_{k+1} \ldots \lambda_n}{\prod_{i=k}^{n} (\lambda_i + 1)} = \frac{\lambda_1 \ldots \lambda_n}{\prod_{i=0}^{n} (\lambda_i + 1)} \frac{\prod_{i=0}^{k-1} (\lambda_i + 1)}{\lambda_1 \ldots \lambda_k}.$$

Thus the matrix is a factorable triangle with

$$a_n = \frac{\lambda_1 \dots \lambda_n}{\prod_{i=0}^{n}(\lambda_i + 1)}, \quad b_k = \frac{\prod_{i=0}^{k-1}(\lambda_i + 1)}{\lambda_1 \dots \lambda_k}.$$

With $u_n := \sum_{k=0}^{n} h_{nk} x_k$,

$$\frac{u_n}{a_n} - \frac{u_{n-1}}{a_{n-1}} = b_n x_n,$$

i.e.,

$$\frac{\prod_{i=0}^{n}(\lambda_i + 1)}{\lambda_1 \dots \lambda_n} u_n - \frac{\prod_{i=0}^{n-1}(\lambda_i + 1)}{\lambda_1 \dots \lambda_{n-1}} = \frac{\prod_{i=0}^{n-1}(\lambda_i + 1)}{\lambda_1 \dots \lambda_n} x_n,$$

or

$$\frac{(\lambda_n + 1)}{\lambda_n} u_n - u_{n-1} = \frac{x_n}{\lambda_n},$$

which gives rise to the following result.

Corollary 6. *Let H denote the H-J Cesáro matrix of order one. Then $x \in c_H$ implies that $x_n = o(\lambda_n)$.*

Corollary 5 can also be proved from Corollary 6 by setting $\lambda_n = n$.

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References

[1] F. Aydin Akgun and B. E. Rhoades, *Factorable generalized Hausdorff matrices*, J. Advanced Math. Studies, 3 (1) (2010), 1–8.

[2] Frank P. Cass and B. E. Rhoades *Mercerian theorems via spectral theory*, Pacific J. Math., 7 (1) (1977), 63–71.

[3] E. Cesáro, Atti d. R. Acad. d Lincei Rend, 4 (4) (1888), 452-457.

[4] G. H. Hardy, *On certain oscillating series*, Quart. J. Math., 38 (1907), 269–288.

[5] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[6] F. Hausdorff, *Summationsmethoden und Momentfolgen, II*, Math. Z., 9 (1921), 280–299.

[7] A. Jakimovski, *The product of summability methods; new classes of transformations and their properties, II*, Note No 4, Contract Number AF61-(052)-187, August, 1959.

[8] Albert Wilansky and Karl Zeller, *The inverse matrix in summability: reversible matrices*, J. London Math. Soc., 32 (1957), 397–408.

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