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FLOQUET THEORY FOR q-DIFFERENCE EQUATIONS

MARTIN BOHNER AND ROTCHANA CHIEOCHAN

Dedicated to Professor Mustafa Kulenović on the occasion of his 60^{th} *birthday*

ABSTRACT. In this paper, we introduce ω-periodic functions in quantum calculus and study the first-order linear q-difference vector equation for which its coefficient matrix function is ω-periodic and regressive. Based on the new definition of periodic functions, we establish Floquet theory in quantum calculus.

1. INTRODUCTION

Floquet theory plays an important rôle in many applications such as in linear dynamic systems with periodic coefficient matrix functions. The study of Floquet theory can be found in Kelley and Peterson [6], Hartman [4], and Cronin [3] for $\mathbb{R}$, and for $\mathbb{Z}$ in Kelley and Peterson [5]. Ahlbrandt and Ridenhour have studied Floquet theory on periodic time scales [1].

In this paper, we are interested to study Floquet theory for q-difference equations, namely dynamic equations on the so-called q-time scale, i.e.,

$$\mathbb{T} := q^{\mathbb{N}_0} := \{q^t : \ t \in \mathbb{N}_0\}, \quad \text{where} \quad q > 1.$$

We present a new definition (see Definition 3.1 below) of periodic functions on the q-time scale and derive some Floquet theory based on the first-order linear equation, called a Floquet q-difference equation,

$$x^{\Delta} = A(t)x, \tag{1.1}$$

where

$$x^{\Delta}(t) := \frac{x(qt) - x(t)}{(q-1)t} \quad \text{for} \quad t \in \mathbb{T},$$

A is an ω-periodic matrix function defined as in Definition 3.1 below, and A also is regressive, i.e.,

$$I + (q-1)tA(t) \quad \text{is invertible for all} \quad t \in \mathbb{T},$$

where I is the identity matrix.

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2. Some auxiliary results

The following definitions and theorems are useful to prove the results in Sections 3 and 4 below.

Definition 2.1. *Let $m, n \in \mathbb{N}_0$ with $m < n$, and $f : q^{\mathbb{N}_0} \to \mathbb{R}$. Then*

$$\int_{q^m}^{q^n} f(t)\Delta t := (q-1)\sum_{k=m}^{n-1} q^k f(q^k).$$

Definition 2.2 (Matrix exponential function)**.** *Let $t_0 \in q^{\mathbb{N}_0}$ and A be an $n \times n$ regressive matrix-valued function on $q^{\mathbb{N}_0}$. The unique matrix-valued solution of the initial value problem*

$$Y^\Delta = A(t)Y, \quad Y(t_0) = I,$$

where I denotes the $n \times n$ identity matrix, is called the matrix exponential *function (at t_0), and it is denoted by $e_A(\cdot, t_0)$.*

For example, if A is an $n \times n$ regressive matrix-valued function on $q^{\mathbb{N}_0}$ and $s = q^m$, $t = q^n$ with $m, n \in \mathbb{N}_0$ and $m < n$, then

$$\begin{aligned} e_A(t,s) &= \prod_{\tau \in q^{\mathbb{N}_0} \cap [s,t)} [I + (q-1)\tau A(\tau)] \\ &= \prod_{k=m}^{n-1} [I + (q-1)q^k A(q^k)], \end{aligned} \tag{2.1}$$

where the matrix product is from the left to the right.

Theorem 2.3 (See [2, Theorem 5.21])**.** *If A is a matrix-valued function on $q^{\mathbb{N}_0}$, then*

(i) $e_0(t,s) \equiv I$ *and* $e_A(t,t) \equiv I$*;*
(ii) $e_A(t,s) = e_A^{-1}(s,t)$*;*
(iii) $e_A(t,s)e_A(s,r) = e_A(t,r)$.

Theorem 2.4 (Liouville's formula [2, Theorem 5.28])**.** *Let A be a 2×2 regressive matrix-valued function on $q^{\mathbb{N}_0}$. Assume that X is a matrix-valued solution of*

$$X^\Delta = A(t)X, \quad t \in q^{\mathbb{N}_0}.$$

Then X satisfies

$$\det X(t) = e_{\mathrm{tr}A + \mu \det A}(t, t_0) \det X(t_0), \quad t \in q^{\mathbb{N}_0},$$

where $\mathrm{tr}A$ *and* $\det A$ *denote the trace and the determinant of A, respectively, and*

$$\mu(t) = (q-1)t, \quad t \in q^{\mathbb{N}_0}.$$

In the last section, we shall show an example of a Floquet q-difference equation, whose coefficient matrix function is defined in terms of trigonometric functions on $q^{\mathbb{N}_0}$.

Definition 2.5 (Trigonometric functions)**.** *Let p be a function defined on $q^{\mathbb{N}_0}$ and suppose $1+(q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$. We define the* trigonometric functions *$\cos_p$ and $\sin_p$ by*

$$\cos_p := \frac{e_{ip} + e_{-ip}}{2} \quad \textit{and} \quad \sin_p := \frac{e_{ip} - e_{-ip}}{2i}.$$

In particular, we have Euler's formula given by

$$e_{ip}(t,t_0) = \cos_p(t,t_0) + i\sin_p(t,t_0),$$

and the identity $[\sin_p(t,t_0)]^2 + [\cos_p(t,t_0)]^2 = 1$ need not hold.

3. Periodic functions

Let $\mathbb{T}$ be a periodic time scale with period $T > 0$, i.e., $t + T \in \mathbb{T}$ whenever $t \in \mathbb{T}$. Then a function $f : \mathbb{T} \to \mathbb{R}$ is called *periodic* if $f(t+T) = f(t)$ for all $t \in \mathbb{T}$. This definition applies for example to the prominent examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. However, $\mathbb{T} = q^{\mathbb{N}_0}$ is not a periodic time scale. Thus we shall introduce the definition of ω-periodic functions on $q^{\mathbb{N}_0}$ as follows.

Definition 3.1. *Let $\omega \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is called* ω-periodic *if*

$$f(t) = q^\omega f(q^\omega t) \quad \textit{for all} \quad t \in q^{\mathbb{N}_0}.$$

A first question concerns the geometrical meaning of ω-periodic functions on $q^{\mathbb{N}_0}$. The following theorem and an example below address this issue.

Theorem 3.2. *Let f be an ω-periodic function on $q^{\mathbb{N}_0}$ and define*

$$c := \int_1^{q^\omega} f(t)\Delta t.$$

Then

$$\int_t^{q^\omega t} f(s)\Delta s = c \quad \textit{for all} \quad t \in q^{\mathbb{N}_0}.$$

Before we prove Theorem 3.2, let us see some examples.

Example 3.3. Let $c \in \mathbb{R}$. We define a function $f : 2^{\mathbb{N}_0} \to \mathbb{R}$ by

$$f(t) := \frac{c}{t} \quad \text{for all} \quad t \in 2^{\mathbb{N}_0}.$$

Then

$$2f(2t) = 2\frac{c}{2t} = \frac{c}{t} = f(t) \quad \text{for all} \quad t \in 2^{\mathbb{N}_0}$$

so, by Definition 3.1, f is 1-periodic. From Definition 2.1, we have

$$\int_1^2 f(t)\Delta t = f(1) = c$$

and

$$\int_t^{2t} f(s)\Delta s = 2^n f(2^n) = t\frac{c}{t} = c, \quad \text{where} \quad t = 2^n.$$

Geometrically, Figure 3.1 shows that the areas under the graph of the function f on the intervals $[2^n, 2^{n+1}]$, $n \in \{0, 1, 2, 3, 4\}$, are all equal to the same constant c.

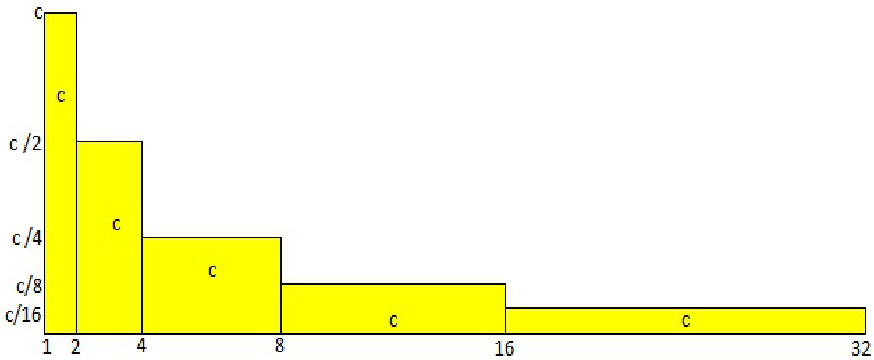

FIGURE 3.1. The constant area of the rectangles corresponding to the 1-periodic function f on the intervals $[2^n, 2^{n+1}]$, $n \in \{0, 1, 2, 3, 4\}$.

Example 3.4. Let $q > 1$ and define $f : q^{\mathbb{N}_0} \to \mathbb{R}$ by

$$f(t) = \frac{1}{t} \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

Then

$$q^\omega f(q^\omega t) = q^\omega \frac{1}{q^\omega t} = \frac{1}{t} = f(t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}$$

so, by Definition 3.1, f is ω-periodic for any $\omega \in \mathbb{N}$. From Definition 2.1, we have that

$$\int_t^{q^\omega t} f(s)\Delta s = (q-1)\sum_{k=n}^{n+\omega-1} q^k f(q^k) = (q-1)\sum_{k=n}^{n+\omega-1} q^k \frac{c}{q^k} = (q-1)\omega c$$

is independent of $t = q^n \in q^{\mathbb{N}_0}$.

Example 3.5. Let $q > 1$ and define $f : q^{\mathbb{N}_0} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{1}{t} & \text{if} \quad \log_q t \quad \text{is odd,} \\ \\ \frac{2}{t} & \text{if} \quad \log_q t \quad \text{is even,} \end{cases}$$

where $\log_q$ is the logarithm to base q, in particular, $\log_q(q^n) = n$ for any $n \in \mathbb{N}_0$. Then

$$\begin{aligned} q^2 f(q^2 t) &= q^2 \begin{cases} \frac{1}{q^2 t} & \text{if} \quad \log_q(q^2 t) \quad \text{is odd,} \\ \\ \frac{2}{q^2 t} & \text{if} \quad \log_q(q^2 t) \quad \text{is even} \end{cases} \\ &= \begin{cases} \frac{1}{t} & \text{if} \quad \log_q t \quad \text{is odd,} \\ \\ \frac{2}{t} & \text{if} \quad \log_q t \quad \text{is even} \end{cases} \\ &= f(t), \end{aligned}$$

so, by Definition 3.1, f is 2-periodic. However, since

$$qf(q) = q\frac{2}{q} = 2 \neq 1 = f(1),$$

f is not 1-periodic.

Proof of Theorem 3.2. We use the principle of mathematical induction to prove that

$$\int_{q^n}^{q^{n+\omega}} f(s)\Delta s = c \tag{3.1}$$

holds for all $n \in \mathbb{N}_0$. From the assumption, we see that (3.1) holds for $n = 0$. Now assume that (3.1) holds for some $n \in \mathbb{N}_0$. Using Definition 2.1, Definition 3.1, again Definition 2.1, and (3.1), we obtain

$$\begin{aligned} \int_{q^{n+1}}^{q^{n+1+\omega}} f(t)\Delta t &= (q-1) \sum_{k=n+1}^{n+\omega} q^k f(q^k) \\ &= (q-1) \left\{ \sum_{k=n+1}^{n+\omega-1} q^k f(q^k) + q^{n+\omega} f(q^{n+\omega}) \right\} \\ &= (q-1) \left\{ \sum_{k=n+1}^{n+\omega-1} q^k f(q^k) + q^n q^\omega f(q^\omega q^n) \right\} \\ &= (q-1) \left\{ \sum_{k=n+1}^{n+\omega-1} q^k f(q^k) + q^n f(q^n) \right\} \end{aligned}$$

$$
\begin{aligned}
&= (q-1)\sum_{k=n}^{n+\omega-1} q^k f(q^k)\\
&= \int_{q^n}^{q^{n+\omega}} f(t)\Delta t\\
&= c.
\end{aligned}
$$

Hence (3.1) holds with n replaced by $n+1$ and the proof is complete. □

Lemma 3.6. *If B is an ω-periodic and regressive matrix-valued function on $q^{\mathbb{N}_0}$, then*

$$e_B(t,s) = e_B(q^\omega t, q^\omega s) \quad \textit{for all} \quad t,s \in q^{\mathbb{N}_0}.$$

Proof. Suppose $s = q^m$ and $t = q^n$ for some $m,n \in \mathbb{N}_0$ with $m < n$. Using (2.1), Definition 3.1, and again (2.1), we obtain

$$
\begin{aligned}
e_B(q^\omega t, q^\omega s) &= e_B(q^{\omega+n}, q^{\omega+m})\\
&= \prod_{k=\omega+m}^{\omega+n-1} \left[I + (q-1)q^k B(q^k)\right]\\
&= \prod_{k=m}^{n-1} \left[I + (q-1)q^{k+\omega} B(q^{k+\omega})\right]\\
&= \prod_{k=m}^{n-1} \left[I + (q-1)q^k q^\omega B(q^\omega q^k)\right]\\
&= \prod_{k=m}^{n-1} \left[I + (q-1)q^k B(q^k)\right]\\
&= e_B(t,s).
\end{aligned}
$$

The proof is complete. □

Theorem 3.7. *Let $t_0 \in q^{\mathbb{N}_0}$ and $\omega \in \mathbb{N}$. If C is a nonsingular $k \times k$ matrix constant, then there exists an ω-periodic regressive matrix-valued function B on $q^{\mathbb{N}_0}$ such that*

$$e_B(q^\omega t_0, t_0) = C.$$

Proof. Let μ_i be the eigenvalues of C, $1 \le i \le k$. For $p \in \{0,1,2,\dots,\omega-2\}$, define

$$
R_p := \begin{pmatrix} J_1 & 0 & \dots & 0\\ 0 & J_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & J_n \end{pmatrix},
$$

where either J_i is the 1×1 matrix $J_i = \mu_i$ or

$$J_i = \begin{pmatrix} \mu_i & 1 & 0 & \dots & 0 \\ 0 & \mu_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mu_i & 1 \\ 0 & \dots & 0 & 0 & \mu_i \end{pmatrix},$$

$1 \le i \le k$, and define

$$R_{\omega-1} := \frac{1}{(q-1)q^{\omega-1}t_0} \left\{ \prod_{k=0}^{\omega-2} \Big[I + (q-1)q^k t_0 R_k \Big]^{-1} C - I \right\},$$

where I is the identity matrix and $\prod_{k=0}^{\omega-2} \big[I + (q-1)q^k t_0 R_k \big]^{-1}$ is the product starting from the right to left. This gives

$$\prod_{k=0}^{\omega-1} \Big[I + (q-1)q^k t_0 R_k \Big] = C,$$

where $\prod_{k=0}^{\omega-1} \big[I + (q-1)q^k t_0 R_k \big]$ is the product starting from the left to right. Moreover, R_p are regressive for all $p \in \{0, 1, 2, \dots, \omega - 1\}$. We define

$$B(q^{\omega m + j} t_0) := \frac{R_j}{q^{\omega} m} \quad \text{for all} \quad j \in \{0, 1, 2, \dots, \omega - 1\} \quad \text{and all} \quad m \in \mathbb{N}_0.$$

Therefore B is ω-periodic and regressive on $q^{\mathbb{N}_0}$ and

$$e_B(q^{\omega} t_0, t_0) = \prod_{k=0}^{\omega-1} \Big[I + (q-1)q^k t_0 B(q^k t_0) \Big] = C,$$

where $\prod_{k=0}^{\omega-1} \big[I + (q-1)q^k t_0 B(q^k t_0) \big]$ is the product starting from the left to right. $\square$

4. Floquet Theory

In this section, we consider the Floquet q-difference equation (1.1), where A is a regressive and ω-periodic matrix-valued function.

Lemma 4.1. *Let $t_0 \in q^{\mathbb{N}_0}$ and suppose x is a solution of the Floquet q-difference equation* (1.1) *satisfying the boundary condition*

$$x(t_0) = q^{\omega} x(q^{\omega} t_0).$$

Then x is ω-periodic.

Proof. Define a function f on $q^{\mathbb{N}_0}$ by

$$f(t) := q^\omega x(q^\omega t) - x(t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

Then $f(t_0) = 0$ and

$$\begin{aligned} f^\Delta(t) &= \frac{f(qt) - f(t)}{(q-1)t} \\ &= \frac{q^\omega x(q^\omega qt) - x(qt) - q^\omega x(q^\omega t) + x(t)}{(q-1)t} \\ &= q^\omega q^\omega \frac{x(qq^\omega t) - x(q^\omega t)}{(q-1)q^\omega t} - \frac{x(qt) - x(t)}{(q-1)t} \\ &= q^\omega q^\omega x^\Delta(q^\omega t) - x^\Delta(t) \\ &= q^\omega q^\omega A(q^\omega t)x(q^\omega t) - A(t)x(t) \\ &= A(t)\left[q^\omega x(q^\omega t) - x(t)\right] \\ &= A(t)f(t). \end{aligned}$$

By unique solvability of the initial value problem $f^\Delta = A(t)f$, $f(t_0) = 0$, we conclude $f(t) = 0$ for all $t \in q^{\mathbb{N}_0}$. By Definition 3.1, x is ω-periodic. $\square$

As usual, we call a matrix-valued function Φ a *fundamental matrix* of the Floquet q-difference equation (1.1) provided it solves (1.1) such that $\Phi(t)$ is nonsingular for all $t \in q^{\mathbb{N}_0}$. The following results gives a representation for any fundamental matrix of the Floquet q-difference equation (1.1).

Theorem 4.2. *Suppose Φ is a fundamental matrix for the Floquet q-difference equation* (1.1)*. Define the matrix-valued function Ψ by*

$$\Psi(t) := q^\omega \Phi(q^\omega t), \quad t \in q^{\mathbb{N}_0}.$$

Then Ψ is also a fundamental matrix for (1.1)*. Furthermore, there exist an ω-periodic and regressive matrix-valued function B and an ω-periodic matrix-valued function P such that*

$$\Phi(t) = P(t)e_B(t, t_0) \quad \textit{for all} \quad t \in q^{\mathbb{N}_0}.$$

Proof. Assume Φ is a fundamental matrix for (1.1) and define Ψ as in the statement of the theorem. Then

$$\begin{aligned} \Psi^\Delta(t) &= \frac{\Psi(qt) - \Psi(t)}{(q-1)t} \\ &= \frac{q^\omega \Phi(q^\omega qt) - q^\omega \Phi(q^\omega t}{(q-1)t} \\ &= q^\omega q^\omega \frac{\Phi(qq^\omega t) - \Phi(q^\omega t)}{(q-1)q^\omega t} \end{aligned}$$

$$\begin{aligned}
&= q^\omega q^\omega \Phi^\Delta(q^\omega t)\\
&= q^\omega q^\omega A(q^\omega t)\Phi(q^\omega t)\\
&= q^\omega A(t)\Phi(q^\omega t)\\
&= A(t)\Psi(t).
\end{aligned}$$

Since $\det\Psi(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$, Ψ is a fundamental matrix for (1.1). Furthermore, define now the nonsingular constant matrix C by

$$C := \Phi^{-1}(t_0)\Psi(t_0).$$

The function D defined by $D(t) = \Psi(t) - \Phi(t)C$, $t \in q^{\mathbb{N}_0}$, satisfies $D(t_0) = 0$ and

$$D^\Delta(t) = \Psi^\Delta(t) - \Phi^\Delta(t)C = A(t)\Psi(t) - A(t)\Phi(t)C = A(t)D(t),$$

and thus, by unique solvability of this initial value problem, we conclude

$$q^\omega \Phi(q^\omega t) = \Psi(t) = \Phi(t)C \quad \text{for all} \quad t \in q^{\mathbb{N}_0}. \tag{4.1}$$

By Theorem 3.7, there exists an ω-periodic and regressive matrix-valued function B such that

$$e_B(q^\omega t_0, t_0) = C. \tag{4.2}$$

Now define the matrix-valued function P by

$$P(t) := \Phi(t)e_B^{-1}(t, t_0), \quad t \in q^{\mathbb{N}_0}.$$

Obviously, P is a nonsingular matrix-valued function on $q^{\mathbb{N}_0}$. Using (4.1), Theorem 2.3 (i), (ii), (4.2), and Lemma 3.6, we obtain

$$\begin{aligned}
q^\omega P(q^\omega t) &= q^\omega \Phi(q^\omega t)e_B^{-1}(q^\omega t, t_0)\\
&= \Phi(t)Ce_B(t_0, q^\omega t)\\
&= \Phi(t)Ce_B(t_0, q^\omega t_0)e_B(q^\omega t_0, q^\omega t)\\
&= \Phi(t)e_B(t_0, t)\\
&= \Phi(t)e_B^{-1}(t, t_0)\\
&= P(t)
\end{aligned}$$

for all $t \in q^{\mathbb{N}_0}$, i.e., P is ω-periodic. □

Theorem 4.3. *Suppose Φ, P, and B are as in Theorem 4.2. Then x solves the Floquet q-difference equation* (1.1) *if and only if y given by $y(t) = P^{-1}(t)x(t)$, $t \in q^{\mathbb{N}_0}$, solves $y^\Delta = B(t)y$.*

Proof. Assume x solves (1.1). Then, as can be seen again by unique solvability of initial value problems as in the proof of Theorem 4.2, we have

$$x(t) = \Phi(t)x_0 \quad \text{for all} \quad t \in q^{\mathbb{N}_0}, \quad \text{where} \quad x_0 := \Phi^{-1}(t_0)x(t_0).$$

Define y by $y(t) = P^{-1}(t)x(t)$, $t \in q^{\mathbb{N}_0}$. Then

$$y(t) = P^{-1}(t)\Phi(t)x_0 = P^{-1}(t)P(t)e_B(t,t_0)x_0 = e_B(t,t_0)x_0,$$

which solves $y^\Delta = B(t)y$. Conversely, assume y solves $y^\Delta = B(t)y$ and define x by $x(t) = P(t)y(t)$, $t \in q^{\mathbb{N}_0}$. Again by unique solvability of initial value problems, we have

$$y(t) = e_B(t,t_0)y_0 \quad \text{for all} \quad t \in q^{\mathbb{N}_0}, \quad \text{where} \quad y_0 := e_B(t_0,t)P(t_0)y(t_0).$$

It follows that

$$x(t) = P(t)y(t) = P(t)e_B(t,t_0)y_0 = \Phi(t)y_0,$$

which solves (1.1). □

Definition 4.4. *Let Φ be a fundamental matrix for the Floquet q-difference equation* (1.1). *The eigenvalues of $q^\omega\Phi^{-1}(1)\Phi(q^\omega)$ are called the* Floquet multipliers *of the Floquet q-difference equation* (1.1).

Remark 4.5. Since fundamental matrices for the Floquet q-difference equation (1.1) are not unique, we shall show that the Floquet multipliers are well defined. Let Φ and Ψ be any fundamental matrices for (1.1) and let

$$C := q^\omega\Phi^{-1}(1)\Phi(q^\omega) \quad \text{and} \quad D := q^\omega\Psi^{-1}(1)\Psi(q^\omega).$$

We show that C and D have the same eigenvalues. Since Φ and Ψ are fundamental matrices of (1.1), we see as in the proof of Theorem 4.2 that there exists a nonsingular constant matrix M such that

$$\Psi(t) = \Phi(t)M \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

It follows that

$$D = q^\omega\Psi^{-1}(1)\Psi(q^\omega) = q^\omega M^{-1}\Phi^{-1}(1)\Phi(q^\omega)M = M^{-1}CM.$$

Therefore C and D are similar matrices, and thus they have the same eigenvalues. Hence, Floquet multipliers are well defined.

Remark 4.6. Note also that the proof of Theorem 4.2 shows that the matrix-valued function

$$q^\omega\Phi^{-1}(t)\Phi(q^\omega t) = \Phi^{-1}(t)\Psi(t) \equiv \Phi^{-1}(1)\Psi(1) = q^w\Phi^{-1}(1)\Phi(q^\omega)$$

does not depend on $t \in q^{\mathbb{N}_0}$, and therefore Floquet multipliers of the Floquet q-difference equation (1.1) are also equal to the eigenvalues of $q^\omega\Phi^{-1}(t)\Phi(q^\omega t)$, where $t \in q^{\mathbb{N}_0}$ is arbitrary.

Theorem 4.7. *The number μ_0 is a Floquet multiplier of the Floquet q-difference equation* (1.1) *if and only if there exists a nontrivial solution x of* (1.1) *such that*

$$q^\omega x(q^\omega t) = \mu_0 x(t) \quad \textit{for all} \quad t \in q^{\mathbb{N}_0}.$$

Proof. Assume μ_0 is a Floquet multiplier of (1.1). Let $t \in q^{\mathbb{N}_0}$. By Remark 4.6, μ_0 is an eigenvalue of $C := q^\omega \Phi^{-1}(t)\Phi(q^\omega t)$, where Φ is a fundamental matrix of (1.1). Let x_0 be an eigenvector corresponding to the eigenvalue μ_0, i.e., we have $Cx_0 = \mu_0 x_0$. Define x by $x(t) = \Phi(t)x_0$ for all $t \in q^{\mathbb{N}_0}$. Then x is a nontrivial solution of (1.1) and

$$q^\omega x(q^\omega t) = q^\omega \Phi(q^\omega t)x_0 = \Phi(t)Cx_0 = \Phi(t)\mu_0 x_0 = \mu_0 x(t).$$

Conversely, assume that there exists a nontrivial solution x of (1.1) such that $q^\omega x(q^\omega t) = \mu_0 x(t)$ for all $t \in q^{\mathbb{N}_0}$. Let Ψ be a fundamental matrix of (1.1). Then $x(t) = \Psi(t)y_0$ for all $t \in q^{\mathbb{N}_0}$ and some nonzero constant vector y_0. Furthermore, $q^\omega \Psi(q^\omega t)$ is a fundamental matrix of (1.1). Hence

$$q^\omega x(q^\omega t) = \mu_0 x(t) \quad \text{and} \quad q^\omega \Psi(q^\omega t)y_0 = \mu_0 \Psi(t)y_0.$$

Since $q^\omega \Psi(q^\omega t) = \Psi(t)D$, where $D := q^\omega \Psi^{-1}(1)\Psi(q^\omega)$ and $\Psi(t)Dy_0 = \Psi(t)\mu_0 y_0$, it follows that $Dy_0 = \mu_0 y_0$, and hence μ_0 is an eigenvalue of D. □

Remark 4.8. By Theorem 4.7, the Floquet q-difference equation (1.1) has an ω-periodic solution if and only if $\mu_0 = 1$ is a Floquet multiplier.

5. Application and an example

Example 5.1. Let p be the 2-periodic function given in Example 3.5 and note that this p is regressive on $q^{\mathbb{N}_0}$. Define

$$A(t) := \begin{pmatrix} 0 & \frac{1}{t}\cos_p(q^2 t, t) \\ \frac{1}{t}\sin_p(q^2 t, t) & 0 \end{pmatrix} \quad \text{for all} \quad t \in q^{\mathbb{N}_0}. \tag{5.1}$$

We apply Lemma 3.6 to show that the coefficient matrix-valued function A is 2-periodic:

$$\begin{aligned} q^2 A(q^2 t) &= q^2 \begin{pmatrix} 0 & \frac{1}{q^2 t}\cos_p(q^4 t, q^2 t) \\ \frac{1}{q^2 t}\sin_p(q^4 t, q^2 t) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{e_{ip}(q^4 t, q^2 t) + e_{-ip}(q^4 t, q^2 t)}{2t} \\ \frac{e_{ip}(q^4 t, q^2 t) - e_{-ip}(q^4 t, q^2 t)}{2ti} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{t}\cos_p(q^2 t, t) \\ \frac{1}{t}\sin_p(q^2 t, t) & 0 \end{pmatrix} \\ &= A(t). \end{aligned}$$

The solution of the Floquet q-difference equation $x^{\Delta} = A(t)x$, where A is defined as in (5.1), satisfying the initial condition $x(1) = x_0$, is $x(t) = e_A(t,1)x_0$, $t \in q^{\mathbb{N}_0}$. If μ_1 and μ_2 are eigenvalues corresponding to the constant matrix

$$C := q^2 e_A^{-1}(1,1)e_A(q^2,1) = q^2 e_A(q^2,1),$$

then by applying Liouville's formula (Theorem 2.4), we get

$$\begin{aligned}\mu_1\mu_2 &= \det C = \det q^2 e_A(q^2,1) = q^4 \det e_A(q^2,1)\\ &= q^4 e_{\mathrm{tr}A+\mu\det A}(q^2,1)\det e_A(1,1)\\ &= q^4 e_f(q^2,1),\end{aligned}$$

where $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$f(t) = \frac{(1-q)\sin_p(q^2t,t)\cos_p(q^2t,t)}{t} \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

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Missouri University of Science and Technology
Department of Mathematics and Statistics
Rolla, Missouri 65409-0020, USA
E–mails: bohner@mst.edu
rckv9@mst.edu