A GENERALIZATION OF CANTOR’S THEOREM

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ABSTRACT. One of the most important results in basic set theory is without doubt Cantor’s Theorem which states that the power set of any set $X$ is strictly bigger than $X$ itself. Specker once stated, without providing a proof, that a generalization is possible: for any natural exponent $m$, there is a natural number $N$ for which if $X$ has at least $N$ distinct elements, then the power set of $X$ is strictly bigger than $X^m$. The aim of this paper is to formalize and prove Specker’s claim and to provide a way to compute the values of $N$ for which the theorem holds.

1. CANTOR AND SPECKER

We state Cantor’s theorem the following way [1]:

**Theorem 1.1.** (Cantor). Let $X$ be a set. There is no injective map $\mathcal{P}(X) \to X$.

This theorem is related to the Generalized Continuum Hypothesis (GCH):

**Hypothesis 1.** Let $X$ and $Y$ be infinite sets. If there are two injective maps $X \to Y$ and $Y \to \mathcal{P}(X)$, then there is a bijection either $X \to Y$ or $Y \to \mathcal{P}(X)$.

In his 1954 article [10], Ernst Specker proves that GCH implies the Axiom of Choice (AC), in the form that for any nonempty set $M$ there exists a function $f : M \to \bigcup M$ such that $f(x) \in x$. The core of the proof lies in the following result:

**Theorem 1.2.** (Specker). Let $X$ be a set. If $X$ has at least five distinct elements, then there is no injective map $\mathcal{P}(X) \to X^2$.

One should note that Specker’s theorem is a “modified version” of Cantor’s theorem with $X^2$ instead of $X$ and a restriction on the number of elements of $X$. In the same article, Specker claims that this theorem can be generalised from the case of exponent 2 to arbitrary finite exponents $m$, without providing a proof of his claim$^1$.

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$^1$He says: “Ein entsprechender Satz gilt fr beliebige endliche Exponenten”; we translate this as: “An analogous theorem holds for arbitrary finite exponents”.

2010 Mathematics Subject Classification. 03E99.

Key words and phrases. Power set, Cantor’s Theorem.
Our aim is to find a function $F : \mathbb{N} \rightarrow \mathbb{N}$ that allows us to state and prove the following:

**Theorem 1.3. (Generalized Cantor).** Let $X$ be a set. For any $m \in \mathbb{N}$, if $X$ has at least $F(m)$ distinct elements, then there is no injective map $\mathcal{P}(X) \rightarrow X^m$. Moreover, if $F(m) \geq 1$ and $X$ has exactly $F(m) - 1$ distinct elements, then there is an injective map $\mathcal{P}(X) \rightarrow X^m$.

Notice that by Cantor’s and Specker’s Theorems we must have $F(1) = 0$ and $F(2) = 5$.

After this brief introduction and some preliminaries, we will define the function $F$ in section 3 in order to prove the main theorem in section 4. After that, we will provide an algorithm to compute $F$ (section 5) and will conclude giving some numerical data (section 6). Throughout this paper we work in Zermelo-Fraenkel Set Theory.

2. **Preliminaries**

Given two sets $X$ and $Y$, as usual we write $X \preceq Y$ to claim the existence of an injective map $X \rightarrow Y$, and $X \cong Y$ to claim the existence of a bijective map $X \rightarrow Y$. It is well-known that $\preceq$ is a non-strict total order, while $\cong$ is an equivalence relation.

**Proposition 2.1.** Let $X$ be a well-ordered infinite set, and let $m > 0$. Then $X^m \cong X$.

**Proof.** We already know this$^3$ for $m = 2$. Suppose that $m = 2^n$ for some $n \in \mathbb{N}$ and prove the theorem in this particular case by induction on $n$.

- $n = 0$: $X \cong X^1 = X^{2^0}$.
- $n \rightarrow n + 1$: $X^{2^{n+1}} = (X^{2^n})^2 \cong X^{2^n} \cong X$.

Trivially, if $m \leq m'$ we get $X^m \preceq X^{m'}$. Thus, given $m$, we can choose $m' > m$ such that $m' = 2^n$ for some $n \in \mathbb{N}$. Then $X \preceq X^m \preceq X^{2^n} \cong X$. We deduce our goal from the Cantor-Schröder-Bernstein theorem. \qed

As usual, denote by $V$ the class of all sets and by $\bigcirc \alpha$ the class of ordinals$^4$. The function

$$\mathcal{H} : V \rightarrow \bigcirc \alpha, X \mapsto \{\alpha \in \bigcirc \alpha : \alpha \preceq X\}$$

is called Hartogs function. The function $\mathcal{H}$ is well-defined, in particular $\mathcal{H}(X)$ is a set whenever $X$ is a set$^5$.

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$^2$If one assumes the Axiom of Choice, a very simple proof of Theorem 1.3 can be given. If $X$ is infinite, the thesis easily follows by proposition 2.1, Zermelo’s well-ordering theorem and Cantor’s theorem; while if $X$ is finite the thesis is a direct consequence of lemma 3.3.

$^3$Theorem 15.11 in Ageron [1]: Let $X$ be an infinite and well-ordered set. Then $X \cong X + 1 \cong 2X \cong X^2$. Tarski further proved [12] that AC is equivalent to a formulation of this theorem without the assumption that $X$ is well-ordered.

$^4$For an introduction to ordinals, see for example Ageron [1], lesson 17.

$^5$Observations 17.2 in Ageron [1]
Given two sets $X$ and $Y$, denote by $\text{Inj}(X,Y)$ the set of injective maps $X \to Y$.

### 3. The Minimal Values Function

For the analytical notions needed in this section, we refer to Davidson & Donsig [3]. Consider the function:

$$f : [\ln 2, \infty) \to ]e^{\ln 2}, \infty), x \mapsto \frac{x}{\ln x} \ln 2$$

Clearly, $f$ is continuous. Its derivative is:

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2} \ln 2 > 0 \iff x > e$$

Thus $f$ is increasing within all its domain. Its infimum and its supremum are easily calculated:

$$\inf f = f(e) = e \ln 2$$

$$\sup f = \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x}{\log_2 x} = +\infty$$

Since $f$ is continuous and increasing, it is invertible. Therefore, the following is well-defined:

$$A : [\ln 2, \infty) \to [e, +\infty), x \mapsto f^{-1}(x)$$

Observe that $e \ln 2 \approx 1.88$. Next, the following is well-defined:

$$F : \mathbb{N} \to \mathbb{N}, m \mapsto \begin{cases} 
1 + |A(m)| & m > 1 \\
0 & m = 1 \\
1 & m = 0 
\end{cases}$$

where

$$\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{N}, x \mapsto \max\{n \in \mathbb{N} : n \leq x\}$$

is the floor function, also known as Gauss’ parentheses.

![Figure 1. Graph of $F$ up to 50.](image)
Similarly, there is the ceiling function
\[
\lceil \cdot \rceil : \mathbb{R} \to \mathbb{N}, x \mapsto \min \{ n \in \mathbb{N} : n \geq x \},
\]
that will be useful later.

We claim that \( F \) is the function we are looking for. To prove so, we need some intermediate results.

**Lemma 3.1.** For any \( m \in \mathbb{N}_{>1} \) we have that \( a = A(m) \) is the minimum value that satisfies
\[
x \in ]a, +\infty[ \Rightarrow 2^x > x^m.
\]

**Proof.** Fixed \( m > 1 \), we want to find the minimum value \( a \in \mathbb{R} \) that satisfies the desired property. Let’s solve the equation \( 2^x = x^m \). Observe that \( A(m) \) is a solution:
\[
x = A(m) \Rightarrow m = f(x) \\
\Rightarrow x^m = \exp(m \ln x) = \exp(f(x) \ln x) = \exp(x \ln 2) = 2^x.
\]

Now we just need to show that \( x > A(m) \) implies \( 2^x > x^m \). Since \( A \) is increasing and \( e \ln 2 < 2 \leq m \) we get that \( e = A(e \ln 2) < A(m) \). Moreover, \( A(m) \leq a \) because \( 2^{A(m)} \nless A(m)^m \). We can thus consider \( x > A(m) \) and obtain:
\[
2^x > x^m \iff m < \frac{x}{\ln x} = f(x).
\]

Since \( f \) is continuous and increasing for \( x > e \) and \( x = A(m) > e \) is a solution, it follows that
\[
\forall x > A(m) : f(x) > f(A(m)) = m,
\]
that is equivalent to say that
\[
\forall x > A(m) : 2^x > x^m.
\]

**Lemma 3.2.** Given \( m \in \mathbb{N}_{>1} \), the equation \( 2^x = x^m \) has exactly one solution \( B(m) \) in \( ]1, e[ \). Moreover, \( B(m) \) satisfies \( A(m) - B(m) \geq 2 \).

**Proof.** Consider \( x \in ]1, e[ \). Since \( x > 0 \) we can write
\[
2^x = x^m \iff m = \frac{x}{\ln x} \ln 2.
\]

Consider the following function which, apart from its domain, is defined as \( f \):
\[
g : ]1, e[ \to \mathbb{R}, x \mapsto \frac{x}{\ln x} \ln 2.
\]

Let’s study \( g \) analogously as we studied \( f \). Clearly, \( g \) is continuous, and its derivative is:
\[
g'(x) = \frac{\ln x - 1}{(\ln x)^2} \ln 2 < 0 \iff x < e.
\]
Thus \( g \) is decreasing within its domain. Its infimum and supremum are:

\[
\inf g = \lim_{x \to e} g(x) = e \log_2 e = e \ln 2,
\]

\[
\sup g = \lim_{x \to 1^+} g(x) = e \log_2 e = +\infty.
\]

Since \( g \) is continuous and decreasing, it is invertible. Therefore the following is well-defined:

\[
B : \left[ e \ln 2, +\infty \right[ \to \left[ 1, e \right[, x \mapsto g^{-1}(x).
\]

As before, \( B \) is continuous and decreasing. It follows, as for \( a = A(m) \), that \( B(m) \) is a solution of the starting equation and that it is unique in \( \left[ 1, e \right[ \). We want an estimate of \( A(m) - B(m) \):

\[
A(m) - B(m) \geq \min_{m \in \mathbb{N}_{> 1}} A(m) - \max_{m \in \mathbb{N}_{> 1}} B(m) = A(2) - B(2).
\]

Let’s verify that \( 4 = A(2) \) and \( 2 = B(2) \):

\[
4 \in \left[ e, +\infty \right[, \quad 2^4 = 16 = 4^2; \quad 2 \in \left[ 1, e \right[, \quad 2^2 = 4 = 2^2.
\]

In conclusion:

\[
A(m) - B(m) \geq A(2) - B(2) = 4 - 2 = 2.
\]

\[ \square \]

**Lemma 3.3.** For any \( m > 1 \), for any \( n \geq F(m) \) we have

\[ 2^n > n^m, \]

while

\[ 2^{F(m)-1} \leq (F(m) - 1)^m. \]

**Proof.** Fix \( m > 1 \). We have:

\[
F(m) = 1 + \lfloor A(m) \rfloor > 1 + A(m) - 1 = A(m).
\]

Then \( F(m) \in \left[ A(m), +\infty \right[ \) and for any \( n \geq F(m) \) we get \( n \in \left[ A(m), +\infty \right[ \). By lemma 3.1 we get \( 2^n > n^m \). Then, by lemma 3.2

\[
B(m) \leq A(m) - 2 < A(m) - 1 < F(m) - 1 \leq A(m).
\]

We have two cases:

1. \( F(m) - 1 \in \left[ e, A(m) \right[ \): for \( x \in \left[ e, A(m) \right[ \) (which is in the domain of \( f \)) we can write

\[ 2^x \leq x^m \iff m \geq f(x). \]

Since \( f \) is continuous and increasing for \( x > e \) and \( x = A(m) > e \) is a solution of \( 2^x = x^m \), it follows that

\[
f(F(m) - 1) \leq f(A(m)) = m,
\]
i.e.
\[2^{F(m) - 1} \leq (F(m) - 1)^m.\]

(2) \(F(m) - 1 \in |B(m)|, e[ : \text{for } x \in |B(m)|, e[ \text{ we get}
\[2^x < x^m \iff m > g(x)\]

(equality is excluded by lemma 3.2). Since \(g\) is continuous and decreasing for \(x < e\) and \(x = B(m) < e\) is a solution of \(2^x = x^m\), it follows that
\[g(F(m) - 1) < g(B(m)) = m,\]
i.e.
\[2^{F(m) - 1} < (F(m) - 1)^m.\] 
\[\square\]

4. A PROOF OF THEOREM 1.3

We first prove separately the case \(m = 0\) of theorem 1.3, verifying that \(F(0) = 1\). Formally:

**Proposition 4.1.** A set \(X\) is nonempty if and only if there is no injective map \(\mathcal{P}(X) \rightarrow \{\emptyset\}\).

**Proof.** The direction \(\Leftarrow\) is easily proved by contraposition since \(\mathcal{P}(\emptyset) = \{\emptyset\}\). To prove \(\Rightarrow\), observe that there is an injective map \(a : \{\emptyset\} \rightarrow X\). Suppose that there is an injective map \(b : \mathcal{P}(X) \rightarrow \{\emptyset\}\). Then there would be an injection \(a \circ b : \mathcal{P}(X) \rightarrow X\), contradicting Cantor’s theorem. \[\square\]

**Lemma 4.2.** Let \(X\) be a set, let \(m \in \mathbb{N}\) and let \(v : \mathcal{P}(X) \rightarrow X^m\) be an injection. Then for any ordinal \(\alpha \geq F(m)\), there is a map \(u_\alpha : \text{Inj}(\alpha, X) \rightarrow X\) such that for any \(i \in \text{Inj}(\alpha, X)\) there is \(u_\alpha(i) \notin i(\alpha)\).

**Proof.** Given \(m\), let \(\alpha\) be an ordinal such that \(\alpha \geq F(m)\). Fix \(i \in \text{Inj}(\alpha, X)\) and set \(I = i(\alpha)\). Let’s build explicitly \(u_\alpha(i) \in X \setminus I\) from \(i\).

1. Suppose that \(\alpha\) is finite. Since \(\alpha\) is a finite ordinal, we can identify it with a natural number \(n \geq F(m)\). Then \(i\) induces a bijection \(n \rightarrow I\); whence \(I\) contains exactly \(n\) elements and \(\mathcal{P}(I)\) contains exactly \(2^n\) elements, as it is well-known. Since \(v\) is injective, \(v(\mathcal{P}(I))\) has \(2^n\) elements too. By lemma 3.3 we get \(|\mathcal{P}(I)| = 2^n > n^m = |I^m|\). It follows that there is \(A \in \mathcal{P}(I)\) such that \(v(A) \notin I^m\). We can write \(v(A) = (x_1, \ldots, x_m)\). Define:
\[u_\alpha(i) := x_k,\]
where \(k = \min \{ j : x_j \notin I^m \}\).

2. Suppose that \(\alpha\) is infinite. Since \(\alpha\) is an infinite ordinal, \(I\) is infinite and well-ordered. Then there is a bijection \(k : I \rightarrow I^m\) (proposition 2.1). Define
\[h : I^m \rightarrow \mathcal{P}(I), c \mapsto \begin{cases} v^{-1}(c) & c \in v(\mathcal{P}(I)) \\ \emptyset & \text{otherwise} \end{cases} .\]
Consider $A := \{ x \in I : x \notin h \circ k (x) \}$. Suppose that $A = h \circ k (x)$ for some $x \in I$. In this case $x \in A \iff x \notin A$, contradiction. Then $A \notin h \circ k (I) = h (I^m)$. It follows that $v (A) \notin I^m$, since otherwise the definition of $h$ would imply $h (v (A)) = A$. Then we can write $v (A) = (x_1, \ldots, x_m)$ and define

$$u_\alpha (i) := x_k$$

where $k = \min\{ j : x_j \notin I^m \}$.

We thus defined $u_\alpha$ for any ordinal $\alpha \geq F(m)$. \hfill \Box

**Proof of theorem 1.3.** Case $m = 1$ is Cantor’s theorem, while $m = 0$ is proposition 4.1. Let $m > 1$. By lemma 3.3 for any $n \geq F(m)$ we have $2^n > n^m$. Suppose that $X$ has at least $F(m)$ distinct elements and that there is an injective map $f : \mathcal{P}(X) \to X^m$. Then there is an injective map $j_{F(m)} : F(m) \to X$. Define by transfinite induction\(^6\) $j_\alpha : \alpha \to X$:

- For $\alpha = F(m)$ we already have $j_{F(m)}$.
- If $j_\alpha$ is defined, define:

$$j_{\alpha+1} (\xi) := \begin{cases} j_\alpha (\xi) & 0 \leq \xi < \alpha \\ u_\alpha (j_\alpha) & \xi = \alpha \end{cases},$$

where $u_\alpha$ is the function defined in lemma 4.2. Since $u_\alpha (j_\alpha) \notin j_\alpha (\alpha)$, we have the injectivity of $j_{\alpha+1}$.
- If $\lambda$ is a limit ordinal and $j_\alpha$ is defined for every $\alpha < \lambda$, define $j_\lambda (\xi) := j_\alpha (\xi)$ where $\xi < \alpha < \lambda$ (such $\alpha$ exists and $j_\alpha$ does not depend on it). Since all $j_\alpha$’s are injective, $j_\lambda$ is injective too.

We obtained that every ordinal $\alpha$ is subpotent to $X$, thus $\mathcal{H}(X) = \emptyset$ but, since $\mathcal{H}(X)$ is a set, this contradicts the Burali-Forti theorem\(^7\). For the second statement, observe that if $X$ has exactly $F(m) - 1$ elements then

$$|\mathcal{P}(X)| = 2^{F(m) - 1} \leq (F(m) - 1)^m = |X^m|.$$

It follows that $\mathcal{P}(X) \not\subseteq X^m$. \hfill \Box

The following result will be useful:

**Proposition 4.3.** $F(3) = 10$.

**Proof.** We have:

$$f(9) \approx 2.8392 < 3 < 3.0103 \approx f(10).$$

Since $f$ is increasing,

$$9 < f^{-1}(3) = A(3) < 10.$$

In conclusion, $F(3) = |A(3)| + 1 = 10$. \hfill \Box

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\(^6\)Principal of transfinite induction, 17.8 in Ageron [1]: Consider a class $H \subseteq \mathbb{N}$ satisfying: (1) $0 \in H$; (2) $\alpha \in H \implies \alpha + 1 \in H$; (3) $\lambda = \sup \{ \alpha \in \lambda \implies \alpha \in H \} \implies \lambda \in H$. Then $H = \mathbb{N}$.

\(^7\)Theorem 17.5(b) in Ageron [1]: The class $\mathbb{N}$ is not a set.
5. An Algorithm for the Function

In this section we want to find an algorithm to compute $F$. Observe that

$$2^x = x^m \iff x = G(x),$$

where

$$G : \mathbb{R}^+ \to \mathbb{R}^+, x \mapsto m \frac{\ln x}{\ln 2}.$$ 

**Lemma 5.1.** Let $m > 0$. The function $G(x)$ is increasing within its domain. Moreover, for $x > m/\ln 2$ we get $|G'(x)| < 1$.

*Proof.* Compute $G'$:

$$G'(x) = \frac{m}{\ln 2} x^{-1}.$$ 

Clearly, for any $x$ in the domain we have $G'(x) > 0$. Also,

$$x > \frac{m}{\ln 2} \iff \frac{m}{\ln 2} x^{-1} < 1.$$ 

Then $0 < G'(x) < 1$. In particular, $|G'(x)| < 1$. □

**Lemma 5.2.** For any $m \in \mathbb{N}, m \geq 2$ we get:

$$\frac{m}{\ln 2} < 6m - 8.$$ 

Moreover, if $m \geq 4$, then $6m - 8 \leq A(m)$.

*Proof.* For the first inequality:

$$\frac{m}{\ln 2} < 6m - 8 \iff m > \frac{8 \ln 2}{6 \ln 2 - 1} \approx 1.76.$$ 

For the second one, prove the following equivalent property:

$$x \geq A(4) \Rightarrow h(x) := 6f(x) - x - 8 \leq 0.$$ 

Compute $h'$:

$$h'(x) = 6f'(x) - 1 = -\frac{(\ln x)^2 - 6 \ln 2 \ln x + 6 \ln 2}{(\ln x)^2} < 0$$

$$\iff \ln x < 3 \ln 2 - \sqrt{3 \ln 2(3 \ln 2 - 2)} \vee \ln x > 3 \ln 2 + \sqrt{3 \ln 2(3 \ln 2 - 2)}$$

$$\iff 0 < x < e^{3 \ln 2 - \sqrt{3 \ln 2(3 \ln 2 - 2)}} \approx 5.33 \vee x > e^{3 \ln 2 + \sqrt{3 \ln 2(3 \ln 2 - 2)}} \approx 12.01.$$ 

In particular, for $x \geq A(4) = 16$ we have that $h$ is decreasing. In addition, $h(16) = 6 \cdot 4 - 16 - 8 = 0$, then $x \geq 16 \Rightarrow h(x) \leq 0$. □

By lemma 5.2, we have that, given $m \geq 4$, the fixed-point method applied to $G$ starting in $x_0 = 6m - 8$ converges to the solution of $2^x = x^m$, i.e. to $x^* = A(m)$. In particular, since $G$ is increasing, given $\overline{x} \in [x_0, x^*[$, we have $\overline{x} < G(\overline{x}) < x^*$. We are not interested in the exact value of $x^*$, but in the one of $F(m) = \lfloor x^* \rfloor + 1$. To this
end we could approximate $x^*$ and then compute $F(m)$, but the following lemma will allow us to build a simpler algorithm.

**Lemma 5.3.** Let $m > 1$.

i) If $A(m) \notin \mathbb{N}$, then $F(m) = \lceil A(m) \rceil$.

ii) If $A(m) \in \mathbb{N}$, then $F(m) = \lceil A(m) \rceil + 1$.

**Proof.**

i) $\lceil A(m) \rceil = \lfloor A(m) \rfloor + 1 = F(m)$.

ii) $\lceil A(m) \rceil = \lfloor A(m) \rfloor \Rightarrow \lceil A(m) \rceil + 1 = \lfloor A(m) \rfloor + 1 = F(m)$. □

Define

$$\tilde{G}: \mathbb{R}^+ \to \mathbb{N}, x \mapsto \lceil G(x) \rceil$$

**Algorithm 1.** Let $m \in \mathbb{N}$ be given in input.

Case 1: If $m = 0$, set $N = 1$.

Case 2: If $m = 1$, set $N = 0$.

Case 3: If $m > 1$:

(a) Set $N_0 := 6m - 8$.

(b) Recursively, while $N_k < \tilde{G}(N_k)$, set $N_{k+1} := \tilde{G}(N_k)$. Denote by $N_n$ the last element in the sequence.

(c) If $N_n = m \log_2 x$, set $N = N_n + 1$, otherwise set $N = N_n$.

Return $N$.

**Proof.** Cases 1 and 2 are Cantor’s theorem and proposition 4.1, respectively. If $m \geq 4$, by lemma 5.2 we can apply the modified version of the fixed-point method. Cases $m = 2$ and $m = 3$ are easily checked by direct computation (using $F(2) = 5$ and $F(3) = 10$). □

![Figure 2. Graph of $F$ up to 50.](image)
6. SOME NUMERICAL DATA

Let's implement algorithm 1 in Matlab/GNU Octave language\(^8\) and write it in figure 3.


```matlab
function N = specker(m)
if m == 0
    N = 1;
elseif m == 1
    N = 0;
else
    g = @(x) ceil(m*log2(x));
    N = 6*m - 8;
    G = g(N);
    while N < G
        N = G;
        G = g(N);
    end
    if N == m*log2(N)
        N = N + 1;
    end
end
end
```

The algorithm 1 is exact. We want to observe numerically its complexity, via the number of iterations and time elapsed. Choose a large interval of values of \(m\), for example from 0 to \(2^{20} = 1048576\) and compute the minimum, the maximum and the average values of \(c\) (number of iterations) and of \(t\) (time elapsed).

\[
\begin{align*}
\text{min}(c) &= 0 & \text{max}(c) &= 9 & \text{sum}(c)/m &= 7.77033 \\
\text{min}(t) &= 1.1921e-05 & \text{max}(t) &= 6.4993e-04 & \text{sum}(t)/m &= 2.8193e-04
\end{align*}
\]

Notice that even for values of \(m\) of the order of \(10^6\) the algorithm terminates in less than ten iterations. Moreover, the elapsed time is also very low, of the order of \(10^{-4}s\) (0.0001s).

Modify the original code in order to show all the iterations. The trace table can be found in figure 1. Observe the following cases:

(1) For \(m \in \{0, 1\}\) we immediately get the final result.
(2) For \(m \in \{2, 4, 32, 4096\}\) we get \(N = N_n + 1^9\).
(3) For all the other values of \(m\) we get \(N = N_n\).

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\(^8\)All the codes in this section (and their respective results) were executed by software Cantor, an interface for Octave.

\(^9\)It can be shown that \(m\) is in this case if and only if \(m = 2^{2k-1} \text{ for some } k \in \mathbb{N}\).
Notice that, even though the number of iterations has an apparent tendency to increase with $m$, this isn’t a strict rule.

**Table 1.** Trace table for some values of $m$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N_0$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$N_4$</th>
<th>$N_5$</th>
<th>$N$</th>
</tr>
</thead>
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<td>0</td>
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**Table 2.** Some values of $F$ computed by the algorithm.

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REFERENCES


(Received: March 21, 2017)
(Revised: July 10, 2018)