SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.12.3.05
Vol.12 (25), No.2, (2016), 317–324, Suppl.

A NOTE ON GENERAL RADICALS OF PARAGRADED RINGS

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Dedicated to the memory of Professor Marc Krasner

Abstract. We discuss the general theory of radicals of paragraded rings, establish that the ADS-Theorem holds, and characterize paragraded normal radicals. It is known that any special radical of a ring can be described by the appropriate class of modules over that ring. In this note we show that all special paragraded radicals of paragraded rings can be described by the appropriate class of their paragraded modules.

1. Introduction

The homogeneous part of the direct product of graded rings needs not to be the direct product of the homogeneous parts of those graded rings, which was the motivation for introducing the notion of a *paragraded ring* [9, 10, 11, 12]. A ring R is called *paragraded* if there exists a mapping $\pi : \Delta \to \mathrm{Sg}(R,+)$, $\pi(\delta) = R_\delta$ $(\delta \in \Delta)$, of a partially ordered set $(\Delta, <)$, which is from below complete semi-lattice and from above inductively ordered, to the set $\mathrm{Sg}(R,+)$ of subgroups of $(R,+)$, called *paragrading*, and the following axioms are satisfied:

$r1)$ $\pi(0) = R_0 = \{0\}$, where $0 = \inf \Delta$; $\delta < \delta' \Rightarrow R_\delta \subseteq R_{\delta'}$;

Remark 1.1. $A = \bigcup_{\delta\in\Delta} R_\delta$ is called *the homogeneous part* of R with respect to π, and elements of A are called *homogeneous elements*.

Remark 1.2. If $x \in A$, we say that $\delta(x) = \inf\{\delta \in \Delta \mid x \in R_\delta\}$ is the *degree of* x. We have $\delta(x) = 0$ if and only if $x = 0$. Elements

2010 *Mathematics Subject Classification.* Primary 16W50; Secondary 16N80.
Key words and phrases. Paragraded rings and modules, Special and normal paragraded radicals.
This paper was presented at the International Scientific Conference *Graded structures in algebra and their applications*, dedicated to the memory of Prof. Marc Krasner, IUC-Dubrovnik, Croatia, September, 22-24, 2016.

$\delta(x)$, $x \in A$, are called *principal degrees* and they form a set which is denoted by Δ_p.

$r2)$ $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} R_\delta = R_{\inf\theta}$;

$r3)$ Homogeneous part A is a generating set of $(R,+)$ with the set of A-inner relations: $x+y=z$;

$r4)$ Let $B \subseteq A$ be a subset such that for all $x, y \in B$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x)$, $x \in B$;

$r5)$ For all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$.

This definition gives a binary operation on Δ, i.e. $\xi\eta = \sup\{\delta(x) \mid x \in R_\xi R_\eta\}$, called the *minimal multiplication* [11, 12], and so, $R_\xi R_\eta \subseteq R_{\xi\eta}$.

An ideal I of a paragraded ring R is called *homogeneous* [12] if I is generated by $I \cap A$ by A-inner relations, where A is the homogeneous part of R.

If R is a paragraded ring with paragrading $\pi : \Delta \ni \delta \mapsto R_\delta$, and M a right R-module, then M is called a *paragraded R-module* [11, 12] if $(M,+)$ satisfies axioms $r1)-r4)$ for π' instead of π, D instead of Δ, M instead of R and $N = \bigcup_{d\in D} M_d$ instead of A, and if moreover, for all $d \in D$ and $\delta \in \Delta$, there exists $t \in D$ such that $\pi'(d)R_\delta \subseteq \pi'(t)$.

In [8], we dealt with some concrete radicals of paragraded rings, namely, with prime and Jacobson radicals. In this note we discuss the general theory of radicals of paragraded rings with emphasis on special radicals.

2. On the general radical theory

In this section we study the general radical theory of the category $\mathcal{K}$ of paragraded rings with quasihomogeneous homomorphisms (see [12]). We follow the well known theory for rings (see e.g. [5]).
Let γ be a class of rings from $\mathcal{K}$ such that

a) γ is closed with respect to quasihomogeneous homomorphisms: if $A \in \gamma$ and B is a homomorphic image of A under a quasihomogeneous homomorphism, then $B \in \gamma$;

b) for every paragraded ring A, the sum $\gamma(A) = \sum(I \lhd A \mid I \in \gamma)$ of homogeneous ideals is in γ;

c) $\gamma(A/\gamma(A)) = 0$ for every paragraded ring A.

Definition 2.1. A class γ of paragraded rings from $\mathcal{K}$, which satisfies $a)$, $b)$ and $c)$, is called a *paragraded radical class* or a *radical class of paragraded rings in the sense of Kurosh and Amitsur* (briefly, *paragraded radical*). $\gamma(A)$ is called a *paragraded γ-radical* of A. A paragraded ring A is called *paragraded γ-radical ring* if $A \in \gamma$, i.e. $\gamma(A) = A$.

One easily proves the following

Theorem 2.2. *A class γ of paragraded rings is a paragraded radical class if and only if*

a) *γ is closed with respect to quasihomogeneous homomorphisms;*
b′) *γ has the inductive property: if $I_1 \subseteq \cdots \subseteq I_\lambda \subseteq \ldots$ is an ascending chain of homogeneous ideals of a paragraded ring A and if $I_\lambda \in \gamma$ for all λ, then $\bigcup_\lambda I_\lambda$ belongs to γ;*
c') *γ is closed under extensions: if I is a homogeneous ideal of a paragraded ring A and if both I and A/I belong to γ, then $A \in \gamma$.*

Lemma 2.3. *Let $A \in \mathcal{K}$, K a homogeneous ideal of I, I a homogeneous ideal of A and a a homogeneous element of A. Then:*

a) $aK + K \lhd I$;
b) $(aK + K)^2 \subseteq K$;
c) the mapping $f : K \to (aK + K)/K$ defined by $f(x) = ax + K$ for any $x \in K$ is a surjective quasihomogeneous homomorphism;
d) $\operatorname{Ker} f \lhd I$.

Proof. As in the case of associative rings, it is straightforward to verify *a*), *b*) and *d*), as well as to prove that the mapping f from *c*) is a surjective homomorphism. Since clearly, a homogeneous element is mapped onto a homogeneous element under f, this mapping is also quasihomogeneous, which concludes the proof. □

As a consequence we have that the ADS-Theorem [1] holds in $\mathcal{K}$, the proof of which is analogous to that for rings (see e.g. [5]).

Theorem 2.4. *For any paragraded radical γ and any $A \in \mathcal{K}$, if I is a homogeneous ideal of A, then $\gamma(I) \lhd A$.*

3. Special radicals

It is known that any radical of a ring can be defined in terms of the appropriate class of modules over that ring [2]. This is also proved for group graded rings [3]. The main aim of this section is to establish that any paragraded radical of a paragraded ring can be described in terms of some class of paragraded modules over that ring, particularly, any special paragraded radical, using methods presented in [2] and [3].

Let A be an associative paragraded ring with a paragrading set Δ. By Σ_A denote a class of paragraded A-modules. By a *paragraded annihilator* $Ann_A(M)$ of a paragraded A-module M we mean a homogeneous ideal of A generated by the set $\bigcup_{\delta\in\Delta}\{x \in A_\delta \mid Mx = 0\}$. In the rest of the article, the paragraded annihilator of a paragraded A-module will be simply called annihilator, since there is no threat of confusion with the classical annihilator.

We define the *kernel of the class* Σ_A to be the set

$$\mathrm{Ker}\Sigma_A = \bigcap_{M \in \Sigma_A} Ann_A(M).$$

If $\Sigma_A = \emptyset$, then we define $\mathrm{Ker}\Sigma_A = A$.

The class Σ, consisting of all paragraded A-modules, is called a *general class of paragraded modules* if the following axioms are satisfied:

P1 If $M \in \Sigma_{A/B}$ for some homogeneous ideal B of A, then $M \in \Sigma_A$;
P2 If $M \in \Sigma_A$ and $B \subseteq Ann_A(M)$, then $M \in \Sigma_{A/B}$;
P3 $\mathrm{Ker}\Sigma_A = 0$ if and only if $\Sigma_B \neq \emptyset$ for all nonzero homogeneous ideals B of A.

Definition 3.1. The Σ-*paragraded radical* $\gamma(\Sigma, A)$ of a paragraded ring A is defined to be the set

$$\gamma(\Sigma, A) = \mathrm{Ker}\Sigma_A,$$

where $\Sigma = \bigcup \Sigma_A$ is a general class of paragraded modules.

Theorem 3.2. *Let Σ be a general class of paragraded modules. Then the Σ-paragraded radical is a radical in the category of paragraded rings. Conversely, if γ is a radical in the category of paragraded rings, then there exists a general class of paragraded modules Σ such that γ coincides with the Σ-radical.*

Proof. The first part can be proved analogously to the proof in the ungraded case (see [2]). The second part is also analogous, but after we establish what the *Dorroh extension of a paragraded ring* is. If A is a paragraded ring with paragrading set Δ, then we define the Dorroh extension A^1 of A as the Dorroh extension of A, regarded as a ring, namely $A^1 = A \times \mathbb{Z}$, with componentwise addition and multiplication $(a, n)(b, m) = (ab + ma + nb, nm)$, but with paragrading defined as follows: $A^1_\delta = \{(a, 0) \mid a \in A_\delta\}$ and $A^1_1 = \{(0, n) \mid n \in \mathbb{Z}\}$. It is easy to verify that A^1 is then a paragraded ring with paragrading set $\Delta \cup \{1\}$, where 1 acts as an identity with respect to the minimal multiplication. □

We turn our attention now to special paragraded radicals. Let $\mathcal{M}$ be a class of paragraded rings. A homogeneous ideal I of a paragraded ring A is called an $\mathcal{M}$-ideal if $A/I \in \mathcal{M}$.

A class $\mathcal{M}$ of paragraded rings is said to be a *special class of paragraded rings* if the following conditions are satisfied:

M1 If B is a homogeneous ideal of a paragraded ring A and P an $\mathcal{M}$-ideal of A which does not contain B, then $P \cap B$ is a proper $\mathcal{M}$-ideal of B;

M2 If Q is a proper $\mathcal{M}$-ideal of B, and if B is a homogeneous ideal of A, then there exists the unique $\mathcal{M}$-ideal P of A such that $P \cap B = Q$.

The following lemma can be proved as in the case of graded rings [3].

Lemma 3.3. *a) Let $\mathcal{M}$ be a special class of paragraded rings, Σ a general class of paragraded modules such that the special radical $\gamma_{\mathcal{M}}$ coincides with Σ-radical. Then a homogeneous ideal P of a paragraded ring A is an $\mathcal{M}$-ideal if and only if $P = Ann_A(M)$ for some paragraded A-module $M \in \Sigma_A$.*
b) The largest special class of paragraded rings coincides with the class of all paragraded prime rings.

Therefore, a class $\mathcal{M}$ of paragraded rings is special if and only if the following axioms are satisfied:

A1 All rings which belong to $\mathcal{M}$ are paragraded prime rings;
A2 If $A \in \mathcal{M}$ and I is a nonzero homogeneous ideal of A, then $I \in \mathcal{M}$;
A3 If B is a homogeneous ideal of a paragraded prime ring A and $B \in \mathcal{M}$, then $A \in \mathcal{M}$.

A class of special paragraded modules which defines the special radical of a given paragraded ring is determined analogously to the ungraded case [2] (see also [5]).

Example 3.4. A paragraded A-module M is called *paragraded prime* if for every nonzero homogeneous submodule N of M and every homogeneous ideal I of A, $NI = 0$ implies $I \subseteq Ann_A(M)$. In [8], the paragraded prime radical of a paragraded ring A is defined as the intersection of all homogeneous prime ideals of A. Now, it is straightforward to verify that the class Σ of all paragraded prime A-modules defines the paragraded prime radical of A.

In [7], the paragraded Jacobson radical of a paragraded ring is defined as the intersection of annihilators of all paragraded irreducible modules over that ring with the same paragrading set. This notion is generalized in [8], namely, the paragraded Jacobson radical (large Jacobson radical) of a paragraded ring A is defined as the intersection of annihilators of all regular (not necessarily regular) paragraded irreducible A-modules.

In [8] it is proved that the large Jacobson radical of a paragraded ring is the largest homogeneous ideal contained in the ordinary Jacobson radical of that ring using the method presented in [6] for graded rings. The same is proved for the prime radical [8] by means known for group graded rings [13]. Here, we give the following theorem.

Theorem 3.5. *Let A be a paragraded ring. Also, let Σ^p be a general class of special paragraded modules which is contained in the class of paragraded irreducible modules and let Σ be the corresponding unparagraded class of*

modules. If $\gamma^p(A)$ is the Σ^p-radical of paragraded ring A and $\gamma(A)$ the Σ-radical of A considered as an ordinary ring, then $\gamma^p(A)$ is the largest homogeneous ideal of A contained in $\gamma(A)$.

Proof. Since every A-module M may be regarded as a paragraded A-module, we have that $\gamma^p(A)$ is contained in the largest homogeneous ideal contained in $\gamma(A)$.

Now, let a be an arbitrary element from the largest homogeneous ideal of A contained in $\gamma(A)$, and let M be a paragraded A-module belonging to Σ^p. Let us prove that $a \in Ann_A(M)$. Suppose $a \notin Ann_A(M)$. Then there exists $x \neq 0$ such that $xa \neq 0$. Since Σ^p is contained in the class of paragraded irreducible modules, M is irreducible. Therefore, xa is a strict generator of M, and hence, $M = xaA$. Let $b \in A$ be such that $x = xab$. Now, since $a \in \gamma(A)$, we also have that $ab \in \gamma(A)$. This means that $\{y - aby \mid y \in A\}$ is not contained in any γ-ideal of A. Hence, $\{y - aby \mid y \in A\}$ is not contained in $Ann_A(M)$. Therefore, there exists $y \in A$ and $k \in Ann_A(M)$ such that $y - aby + k = -ab$. Now, $0 = x - xab - (x - xab)y = x - x(ab + y - aby) = x + xk$, which implies $x = 0$, a contradiction. Hence, a is contained in $\gamma^p(A)$. $\square$

4. Paragraded normal radicals

In [4] group-graded normal radicals were observed in order to generalize results from [14]. To do something similar in the case of paragraded rings, that is, in order to define *paragraded normal radicals*, we need some preliminary notions.

Definition 4.1. If A is a paragraded ring, a paragraded radical γ is called *paragraded left strong* if $I \in \gamma$ implies $I \subseteq \gamma(A)$ for all left ideals I of A.

The notion of a *paragraded right strong radical* is defined symmetrically.

Let A and B be paragraded rings with paragrading sets Δ and Δ', respectively. Also, let V be a paragraded $A - B$-bimodule and W a paragraded $B - A$-bimodule with paragrading sets D and D', respectively.

Definition 4.2. A Morita context (A, V, W, B) is called a *paragraded Morita context* if:

$$(\forall d \in D)(\forall d' \in D')(\exists \delta \in \Delta)\ V_d W_{d'} \subseteq A_\delta$$

and

$$(\forall d \in D)(\forall d' \in D')(\exists \delta' \in \Delta')\ W_{d'} V_d \subseteq B_{\delta'}.$$

Theorem 4.3. *Let (A, V, W, B) be a Morita context. Then $R = \left(\begin{smallmatrix} A & V \\ W & B \end{smallmatrix}\right)$ is a paragraded ring. In particular, if a given Morita context is paragraded, R is a paragraded ring.*

Proof. We may use the proof of Theorem 2.1 from [8] and make R paragraded in a similar manner. We take $R_1 = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $R_2 = \left(\begin{smallmatrix} A & V \\ 0 & 0 \end{smallmatrix}\right)$, $R_3 = \left(\begin{smallmatrix} A & 0 \\ W & 0 \end{smallmatrix}\right)$, $R_4 = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, and so on. □

Definition 4.4. A radical γ of paragraded rings is called *paragraded normal* if

$$V\gamma(B)W \subseteq \gamma(A)$$

for every paragraded Morita context (A, V, W, B).

Using the *paragraded Dorroh extension* from the proof of Theorem 3.2, the following result can be proved the same way as in the case of rings [14].

Proposition 4.5. *If γ is a paragraded normal radical, then γ is left and right strong.*

We say that a paragraded radical γ is *paragraded principally left hereditary* if $A \in \gamma$ implies $Aa \in \gamma$ for every homogeneous element $a \in A$. It is clear how to define the notion of *paragraded principally right hereditary* radical.

Proposition 4.6. *Every paragraded normal radical γ is paragraded principally left hereditary and principally right hereditary.*

Proof. Let $A \in \gamma$ and let $a \in A$ be its homogeneous element. As in the case of rings we observe the paragraded Morita context (Aa, A^1, Aa, A) and establish that $A^2a \subseteq \gamma(Aa)$. The mapping $f : A \to Aa/\gamma(Aa)$ given by $f(x) = xa + \gamma(Aa)$ for all $x \in A$ is a quasihomogeneous homomorphism, and the rest of the proof is similar to the proof given in the case of rings. □

Finally, taking into account previous results, with the appropriate modification of a proof given in [14] for rings, the following result holds.

Theorem 4.7. *A paragraded radical γ is paragraded normal if and only if γ is paragraded left strong and paragraded principally left hereditary (or paragraded right strong and paragraded principally right hereditary).*

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(Received: July 26, 2016)
(Revised: September 15, 2016)

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