ON \((n,m)\)-GROUPS FOR \(n > 2m\)

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Abstract. In this article a theorem about a \((2m,m)\)-group of Čupona-Dimovski is generalized.

1. Preliminaries

Definition 1.1. [1] Let \(n \geq m + 1\) and let \((Q; A)\) be an \((n,m)\)-groupoid \((A : Q^n \to Q^m)\). We say that \((Q; A)\) is an \((n,m)\)-group iff the following statements hold:

(i) For every \(i, j \in \{1, \ldots, n - m + 1\}\), \(i < j\), the following law holds

\[
A(x_i^{i-1}, A(x_i^{i+n-1}, x_{i+n}^{2n-m})) = A(x_j^{j-1}, A(x_j^{j+n-1}, x_{j+n}^{2n-m}))
\]

\([< i, j > \text{ - associative law}]\); and

(ii) For every \(i \in \{1, \ldots, n - m + 1\}\) and for every \(a_i^m \in Q\) there is exactly one \(x_i^m \in Q^m\) such that the following equality holds

\[
A(a_i^{i-1}, x_i^m, a_i^{n-m}) = a_i^{n-m+1}.
\]

Also see [3].

Definition 1.2. [6] Let \(n \geq 2m\) and let \((Q; A)\) be an \((n,m)\)-groupoid. Let also \(e\) be a mapping of the set \(Q^{n-2m}\) into the set \(Q^m\). Then, we say that \(e\) is a \(\{1, n - m + 1\}\)-neutral operation of the \((n,m)\)-groupoid \((Q; A)\) iff for every sequence \(a_i^{n-2m}\) over \(Q\) and for every \(x_i^m \in Q^m\) the following equalities hold

\[
A(x_i^m, a_i^{n-2m}, e(a_i^{n-2m})) = x_i^m \quad \text{and} \quad A(e(a_i^{n-2m}), a_i^{n-2m}, x_i^m) = x_i^m.
\]

Remark. For \(m = 1\) \(e\) is a \(\{1, n\}\)-neutral operation of the \(n\)-groupoid \((Q; A)\) [5]. Cf. Chapter II in [8].

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Proposition 1.3. [6] Let \((Q; A)\) be an \((n, m)\)–groupoid and let \(n \geq 2m\).
Then there is at most one \([1, n - m + 1]\)–neutral operation of \((Q; A)\).

Proposition 1.4. [6] Every \((n, m)\)–group \((n \geq 2m)\) has a \([1, n - m + 1]\)–neutral operation.

Also see [7].

2. Auxiliary proposition

Proposition 2.1. [4] Let \((Q; A)\) be an \((n, m)\)–group, e its \([1, n - m + 1]\)–neutral operation (1.2-1.4) and \(n > 2m\). Then for every \(a_1^{n-2m}, x_1^m \in Q\) and for all \(j \in \{1, \ldots, n - 2m + 1\}\) the following equalities hold

\[
A(x_1^m, a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}) = x_1^m \quad \text{and} \quad A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, x_1^m) = x_1^m.
\]

Remark. For \(m = 1\) see Proposition 1.1–IV in [8].

Main part of the proof.

\[
F(x_1^m, a_1^{n-2m}) \overset{\text{def}}{=} A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, x_1^m) \Rightarrow
A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, F(x_1^m, a_1^{n-2m})) =
A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, x_1^m)) \overset{(i)}{=} 
A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, F(x_1^m, a_1^{n-2m})) =
A(a_j^{n-2m}, A(e(a_1^{n-2m}), a_1^{n-2m}, e(a_1^{n-2m})), a_1^{j-1}, x_1^m) \overset{1.2.1.4}{=} 
A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, F(x_1^m, a_1^{n-2m})) =
A(a_j^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, x_1^m) \overset{(ii)}{=} F(x_1^m, a_1^{n-2m}) = x_1^m.
\]

Whence, we obtain (2). \(\square\)

3. Result

Theorem 3.1. Let \(n > 2m, m > 1\), \((Q; A)\) be an \((n, m)\)–group and e its \([1, n - m + 1]\)–neutral operation (1.2-1.4). Then for all \(i \in \{0, 1, \ldots, m\}\) for all \(t \in \{1, \ldots, n - 2m + 1\}\), for every \(x_1^m \in Q^m\) and for every sequence \(a_1^{n-2m}\) over \(Q\) the following equality holds

\[
A(x_1^i, a_t^{n-2m}, e(a_1^{n-2m}), a_1^{j-1}, x_1^{m}) = x_1^m.
\]

Remark. Theorem 3.1 for \(n = 2m\) \((m > 1)\) is proved in [2]. Also see [3].
Main part of the proof.

1) Instead of \( e(a_1^{n-2m}) \) we write

\[
e_j(a_1^{n-2m}) \mid_{j=1}^m
\]

where \( e_j : Q^{n-2m} \to Q \).

2)

\[
A(x_i^t, a_t^{n-2m}, e(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = \quad (2)_{j=1}^{m+1}
\]

\[
A(a_1^{n-2m}, e(a_1^{n-2m}), A(x_i^t, a_t^{n-2m}, e(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m)) = \quad (1)
\]

\[
A(x_i^t, a_t^{n-2m}, e(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = \quad (i)
\]

\[
A(a_1^{n-2m}, e_j(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = \quad (i)
\]

\[
A(a_1^{n-2m}, a_t^{n-2m}, e(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = \quad (1)
\]

\[
A(a_1^{n-2m}, e_j(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = \quad (i)
\]

\[
A(a_1^{n-2m}, e(a_1^{n-2m}), a_1^{t-1}, x_{i+1}^m) = \quad (2)_{j=1}^{m+1}
\]

\[
x_1^m, \quad 0 < i < m.
\]

\[\square\]

REFERENCES
